Dynamic Utilities

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Towards a constitutive analogue of the Black and Scholes theory in incomplete markets
Modelling, optimal behavior, valuation and risk management

- Market prices of underlying securities — semimartingales
- Risk preference formulation — utility payoffs
- Specification of admissible strategies — state and control constraints
- Construction of optimal strategies — optimal stochastic control
- Valuation — linear (complete) or nonlinear (incomplete) expectations

In complete markets only the first and last step are important
Toy example

• **Market:** stock, bond \((r = 0)\)

\[
dS_t = S_t(\mu_t \, dt + \sigma_t \, dW_t)
\]

\(W\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\)

\(\mu_t, \sigma_t \in \mathcal{F}_t; \quad \mathcal{F}_t = \sigma(W_u; 0 \leq u \leq t)\)

• **Wealth:** \(X_t = \pi^0_t + \pi_t\)

\[
dx_t = \pi_t(\mu_t \, dt + \sigma_t \, dW_t)
\]

• **Utility payoff:** \(U\) deterministic function of wealth (increasing, convex, concave)

\[
\gamma = -\frac{U''(x)}{U'(x)} \quad \text{risk aversion}
\]

• **Criterion:** For the investment horizon \([t, T]\)

\[
V(X_t, t) = \sup_{\mathcal{A}} E_{\mathbb{P}}(U(X_T) / \mathcal{F}_t)
\]
• **Liability:** \( C_T \in \mathcal{F}_T \), \( \hat{T} \leq T \)

• **Asset and liability management**

\[
V^{CT}(X_t, t) = \sup_{A} E_{\mathbb{P}}(U(X_T - C_T) / \mathcal{F}_t)
\]

• **Valuation:** Find indifference value \( \nu_t(C_T) \in \mathcal{F}_t \) that satisfies

\[
V(X_t, t) = V^{CT}(X_t + \nu_t(C_T), t)
\]
Fundamental properties of a utility-based pricing system

- Semigroup property of prices

\[ \nu_t(C_T; \gamma) = \nu_t(\nu_s(C_T; \gamma); \gamma) \]

- Numeraire invariance

\[ \frac{\nu_t^Y(C_T; \gamma^Y)}{Y_t} = \nu_t^Z \left( \frac{C_T}{Y_T}; \gamma^Z \right) \]

\[ \gamma^{Y,Z} : \text{risk aversion w.r.t. } Y, Z \text{ units} \]

Traditional utility functionals fail to produce rich classes of prices
Outline

Motivational examples from indifference valuation

- Numeraire invariance
  - Stochastic risk aversion
- Semigroup property of prices
  - Invariance of optimal utilities across trading horizons

Stochastic utilities

- Normalization
  - Evolution
  - Backward
  - Forward
“Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model”
Indifference Pricing, PUP (2005, Musiela-Z.)

“Investment and valuation under backward and forward dynamic utilities in a stochastic factor model”
Preprint (2005, Musiela-Z.)

“Numeraire consistency, stochastic risk preferences and indifference valuation”
The term structure of preferences
Fundamental questions

• What is the proper specification of the investors’ risk preferences?

• Are risk preferences static or dynamic?

• Are they affected by the market environment and the trading horizon?

• Are there endogenous structural conditions on risk preferences?

• How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?
Requirements for a well posed indifference pricing system

(work in progress MZ, ZZ)

Risk preferences need to satisfy structural conditions across units, trading horizons and maturities

↓

Self-generating stochastic utilities

Martingality of risk tolerance process, \( \delta_t = -\frac{U'_t(x)}{U''_t(x)} \), w.r.t. to the appropriate risk-neutral measure, is one of the requirements we observed so far
Self-generating stochastic utilities

Utility process:

\[ U_s(\omega) \quad U_t(\omega) \quad U_T(\omega) \quad U_t \in \mathcal{F}_t \]

Value function process:

\[ V_s(\omega) \quad V_t(\omega) \quad V_T(\omega) \quad V_t \in \mathcal{F}_t \]

\[ V_s(X_s; T) = \sup_{A} E_{\mathbb{P}}[U_t(X_t; T)/\mathcal{F}_s] \]

Requirement for correct pricing

\[ V_s = U_s \quad V_t = U_t \quad V_T = U_T \]

Alignment of current utility with its associated value function
Models of the term structure of risk preferences

• Normalization point
• Evolution equation

Questions

• Are dynamic risk preferences and market behavior interlinked?
• How is the utility evolving w.r.t. time?

Answers

• Preferences and market cannot be defined in isolation
• Utilities are defined backward and forward in time
Backward stochastic utilities

- Traded asset: \( S_t, \ t \in [0, T] \), Non-traded asset: \( Y_t, \ t \in [0, T] \)

Filtrations: \( \mathcal{F}_t^S, \mathcal{F}_t^Y, \mathcal{F}_t^{(S,Y)} \)

- Normalization point: \( T > 0 \)

- The backward, normalized at \( T > 0 \), dynamic utility \( U_t^B(x; T) \) is defined as an \( \mathcal{F}_t^{(S,Y)} \)-measurable process solving

\[
U_s^B(X_s; T) = \sup_{\alpha} E_{\mathbb{P}} \left( U_t^B(X_t; T) \middle| \mathcal{F}_s^{(S,Y)} \right)
\]

\[
X_t = X_s + \int_s^t \alpha_u dS_u
\]

\[
U^B(x, T; T) = U(x; \gamma_T(\omega)) \ ; \ U \text{ given}
\]
Backward stochastic utilities

\[ U^B_s(\omega) \quad U^B_t(\omega) \quad U^B_T(\omega) \]

\[ s \quad t \quad T \]

utility generation backwards in time

\[ U^B_s(X_s; T) = \sup_{\mathcal{A}} E[U^B_t(X_t; T) / \mathcal{F}_s] \]

\[ = \sup_{\mathcal{A}} E[U^B(X_T; T) / \mathcal{F}_s] = V_s(X_s; T) \]

- Self-generation and invariance w.r.t. trading horizons
- Backward utility is, essentially, the traditional value function but with random terminal criterion
- Backward utility and market behavior interlinked
Complete market and stochastic risk tolerance

**Example**

\[ U_T^B = -e^{-\gamma T x} \]

\[ \gamma_T \in \mathcal{F}_T \]

\[ U_s^B(X_s; T) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left[ U_t^B(X_t; T)/\mathcal{F}_s \right] = -e^{-\frac{X_s}{\delta_s} - \tilde{H}_s} \]

\[ \delta_s = E_{\mathbb{Q}} \left( \frac{1}{\gamma_T}/\mathcal{F}_s \right) \]

\[ \tilde{H}_s = E_{\tilde{\mathbb{Q}}} \left[ \int_s^T \frac{1}{2} \lambda_u^2 du/\mathcal{F}_s \right] \]

\[ S_s = E_{\mathbb{Q}}(S_T/\mathcal{F}_s) \]

\[ \frac{1}{\delta_s} S_s = E_{\tilde{\mathbb{Q}}} \left( \frac{1}{\delta_T} S_T/\mathcal{F}_s \right) \]

\[ \mathbb{Q}, \tilde{\mathbb{Q}} \] risk-neutral measures
Forward stochastic utility (MZ 2003, 2005)

- How do we decide today about our future risk attitude?

- How do we valuate, via utility, claims of arbitrary maturities?

\[ T \]

\[ t \quad C_s, \ldots, C_T \]

\( T \)

\( t \quad \text{new claim arrives} \quad \text{longer maturity} \)

valuation already done w.r.t. normalization point \( T \)
Forward stochastic utility (con’)

- Normalization point $s > 0$

- Dynamic utility is specified going forward in time, $t \geq s$

- No need to have a fixed trading horizon
Forward stochastic utility

- Normalization point: $s > 0$

- $U_t^F(x; s), s > 0, t \geq s$ is defined as an $\mathcal{F}_t^{(S,Y)}$-measurable process solving

$$U_t^F(X_t; s) = \sup_{\alpha} E_P(U_{t'}^F(X_{t'}; s)|\mathcal{F}_t^{(S,Y)})$$

$$X_{t'} = X_t + \int_t^{t'} \alpha_u dS_u$$

$$U_s^F(x) = U(x; \omega) ; \quad U \text{ given}$$
**Forward stochastic utilities**

\[ U^F_s(x; s) \in \mathcal{F}_s \]

utility generation forward in time

\[ U^F_t(X_t; s) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left[ U^F_{t'}(X_{t'}; s)/\mathcal{F}_t \right] = V^F_t(X_t; s) \]

Self-generation and invariance w.r.t. trading horizon
Complete market case and stochastic Sharpe ratio

Example

\[ U_F^S(x; s) = -e^{-\gamma x} - e^{-\gamma x + \int_s^t \frac{1}{2} \lambda_u^2 du} - e^{-\gamma x + \int_s^{t'} \frac{1}{2} \lambda_u^2 du} \]

0 \quad s \quad t \quad t'

Self-generation holds

\[ -e^{-\gamma x + \int_s^t \frac{1}{2} \lambda_u^2 du} = \sup_A E_Prob \left[ -e^{-\gamma X_{t'} + \int_t^{t'} \frac{1}{2} \lambda_u^2 du} / \mathcal{F}_t \right] \]
Forward versus backward utilities

- **Backward** stochastic utilities aggregate market information while **forward** stochastic utilities use information revealed dynamically by the market.

- **Forward** indifference prices do not depend on the preference normalization point.

- Forward indifference prices are represented in a **more intuitive** manner.
Open questions

• When do backward stochastic utilities exist? Are they unique? Are they robust w.r.t. terminal risk preference specification? (relatively easy)

• When do forward stochastic utilities exist? Are they unique? Are they robust w.r.t. initial risk preference specification? (very hard)

• What is the term structure of the risk tolerance process in backward and forward setting?

• How is indifference valuation built in terms of backward and forward preferences?

• Are backward and forward indifference prices equal?

• How do they depend on the risk preference normalization point?
Utility-based backward and forward pricing systems for stochastic volatility models
A simple diffusion case

- **Market dynamics**

\[
dS_s = \mu(Y_s, s)S_s \, ds + \sigma(Y_s, s)S_s \, dW^1_s \\
\]

\[
dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \\
\]

\[
\rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s = \lambda(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)}
\]

- **Minimal relative entropy measure:**

\[
\frac{dQ}{dP} = \exp\left(-\int_0^T \lambda_s \, dW^1_s - \int_0^T \lambda_s^1 \, dW^1_s - \frac{1}{2} \int_0^T (\lambda^2_s + (\lambda^\perp_s)^2) \, ds\right)
\]

\[
\lambda^\perp_s = \lambda^\perp_s(Y_s, s); \quad \lambda^\perp(y, t) \sim \text{gradient of the sln of a quasilinear pde}
\]

(Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)
Backward stochastic utilities

For $T > 0$ the process $\{U_t^B(x; T) : 0 \leq t \leq T\}$ defined, for $x \in \mathbb{R}$, by

$$U_t^B(x; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x} - H_T(Q(\cdot|\mathcal{F}_t)|\mathbb{P}(\cdot|\mathcal{F}_t)) & \text{if } 0 \leq t < T
\end{cases}$$

with

$$H_T(Q(\cdot|\mathcal{F}_t)|\mathbb{P}(\cdot|\mathcal{F}_t)) = E_Q \left( \int_t^T \frac{1}{2} (\lambda_s^2 + \lambda_s^\perp)^2 d\xi |\mathcal{F}_t) \right)$$

is the, normalized at time $T$, backward stochastic exponential utility.
Forward stochastic utilities \hspace{1cm} (MZ 2005)

For $T \geq s \geq 0$ and $x \in \mathbb{R}$, the process $U^F_t(x; s) \in \mathcal{F}_t^{(S,Y)}$ defined by

$$U^F_t(x; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x + h_t} & \text{if } t \geq s
\end{cases}$$

with

$$h_t = \int_s^t \frac{1}{2} \lambda_u^2 \, du$$

is the forward, normalized at time $s$, stochastic exponential utility.
Backward and forward indifference prices

- Backward price

\[
 h^B_t (C_{\hat{T}}; T) = \mathcal{E}^{(t,\hat{T})}_{Q^{me}} (C_{\hat{T}}; T)
\]

\(\hat{T} \leq T\), \(Q^{me}\) : minimal entropy measure

- Forward price

\[
 h^F_t (C_{\hat{T}}; s) = \mathcal{E}^{(t,\hat{T})}_{Q^{mm}} (C_{\hat{T}})
\]

No dependence on \(s\), \(Q^{mm}\) : minimal martingale measure

Local description of \(h^B(x, t)\) and \(h^F(x, t)\) via quasilinear pdes with quadratic nonlinearities
Forward utility processes
Construction of forward utility process via forward mollifiers  
(MZ (2005, 2006))

- Recall that the forward utility process $U_t^F(x; s) \in \mathcal{F}_t$ is defined as

\[
\begin{align*}
U_t^F(x; s) &= \sup_{A} E_\mathbb{P}(U_{t'}^F(x + \int_t^{t'} \alpha_u dS_u; s) / \mathcal{F}_t) \\
U_s(x; s) &\in \mathcal{F}_s
\end{align*}
\]

- The process $U_{t'}^F(X_{t'}; s)$ is an $\mathcal{F}_{t'}$-supermartingale w.r.t. $\mathbb{P}$ and a martingale at the optimum wealth $X_{t'}^*$,

\[
X_{t'}^* = X_{t}^* + \int_t^{t'} \alpha_u^* dS_u
\]

- For fixed $x$, $U_{t'}^F(x; s) \leq U_t^F(x; s)$, $t' \leq t$ (impatience)

- For fixed $x$, $U_t^F(x; s)$ is increasing and concave w.r.t. the wealth variable
• Utility molifier: \( u(x, t) \)

For simplicity, we consider a Brownian filtration market environment

\[
dX_t = \beta_t(\lambda_t \, dt + dW_t) \quad \text{(wealth process)}
\]

\[
dA_t = \frac{1}{2} \lambda_t^2 \, dt \quad \text{(internal “clock” of the market price of risk)}
\]

\[M_t = u(X_t, A_t)\]

Stochastic calculus yields that \( M_t \) is a supermartingale if the matrix

\[
D = \begin{pmatrix}
  u_t(x, t) & u_x(x, t) \\
  u_x(x, t) & u_{xx}(x, t)
\end{pmatrix}
\]

is negative definite
• An “ill-posed” nonlinear pde

\[
\begin{cases}
    u_t u_{xx} = u_x^2 \\
    u(x, 0) = U(x, 0)
\end{cases}
\]

\( U \) : forward utility assigned at normalization point

• Level sets of utility mollifier

\[
\begin{cases}
    u_t + r(x, t)u_x = 0 \\
    u(x, 0) = U(x, 0)
\end{cases}
\]
• Variational results yield that the slope $r$ solves the autonomous equation

$$\begin{cases} 
    r_t + r^2 r_{xx} = 0 \\
    r(x, 0) = -\frac{U'(x, 0)}{U''(x, 0)} 
\end{cases}$$

• The local risk aversion $\gamma(x, t) = -\frac{U_{xx}(x, t)}{U_x(x, t)}$ satisfies

$$\begin{cases} 
    \gamma_t = r_{xx} \\
    \gamma(x, 0) = \frac{1}{r(x, 0)} 
\end{cases}$$

• The $(u - r)$ system yields a forward utility constitutive law
The \((u - r)\) utility constitutive law

\[
\begin{aligned}
  rt + r^2rx &= 0 \\
u_t + ru_x &= 0
\end{aligned}
\]

with

\[
u(x, 0) = u_0(x) \quad \text{and} \quad r(x, 0) = -\frac{u'_0(x)}{u''_0(x)}, \quad t > 0, \ x \in \mathbb{D}.
\]

- Inverse problem and transport equation
- Slope of characteristics given by local risk tolerance
Construction of forward utilities

- Investment universe
  \[ dB_t = r_t B_t \, dt \]
  \[ dS^i_t = S^i_t (\mu^i_t \, dt + \sum_{j=1}^{d} \sigma^j_t \, dW^j_t) \]

  \( W = (W^1, \ldots, W^d) \) \( d \)-dim. standard Brownian motion

  \( \mu_t, \sigma_t \in \mathcal{F}_t^W \), \( \mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t \)

- Wealth process
  \[ dX_t = \beta_t \cdot (\lambda_t \, dt + dW_t) \]
  \[ \beta_t = \sum_{i=1}^{k} \frac{\pi^i_t \sigma^i_t}{B_t} \]
• Benchmark process

\[ dY_t = Y_t \delta_t \cdot (\kappa_t \, dt + dW_t) ; \quad \kappa_t, \delta_t \in \mathcal{F}_t^W \]
\[ Y_0 = y > 0 \]

• Internal market time change

\[ dZ_t = \eta_t \, dt ; \quad \eta_t \in \mathcal{F}_t^W \]
Forward utility process

- $u(x, t)$ solution of the $(u - r)$ utility law

- $X, Y, Z$ Itô processes representing wealth, benchmark, internal market clock

- Forward utility

$$U_t^F(\omega) = u\left(\frac{X_t}{Y_t}, Z_t\right)$$
Conclusions

• Going forward in investment time...

• \((u - r)\) utility evolution system

• Inverse problem + transport equation

• Construction of forward utility process via the mollifier \(u\) and the \((X, Y, Z)\) market processes

• Optimal policies turn out to have a very intuitive structure as functions of market parameters, risk tolerance, benchmark performance, internal market clock
Exponential case

\[ U_t = \exp \left( -\frac{X_t}{Y_t} + Z_t \right) \]

- Market input

\[ \mathcal{E}_t = \exp \int_0^t \delta_t \cdot (\lambda_s ds + dW_s) \]

- Optimal wealth

\[ X_t^* = \mathcal{E}_t \left( X_s + \int_s^t \mathcal{E}^{-1}_u Y_u (\sigma_u \sigma_u^+ \lambda_u - \delta_u) \cdot ((\lambda_u - \delta_u)du + dW_u) \right) \]

- Optimal portfolios

\[ \pi_t^* = X_t^* \mathcal{E}_t B_t \sigma_t^+ \delta_t + B_t Y_t \sigma_t^+ (\lambda_t - \delta_t) \]

\[ + B_t \mathcal{E}_t \int_s^t \mathcal{E}^{-1}_u Y_u (\sigma_u \sigma_u^+ \lambda_u - \delta_u) \sigma_u^+ \delta_u ((\lambda_u - \delta_u)du + dW_u) \]