

## Part II

# CONTINUOUS TIME STOCHASTIC PROCESSES

# Chapter 4

For an advanced analysis of the properties of the Wiener process, see:

Revus D. and Yor M.: *Continuous martingales and Brownian Motion*

Karatzas I. and Shreve S. E.: *Brownian Motion and Stochastic Calculus*

Beginning from this lecture, we study continuous time processes. A stochastic process  $X$  is defined, in the same way as in Lecture 1, as a family of random variables  $X = \{X_t : t \in T\}$  but now  $T = [0, \infty)$  or  $T = [a, b] \subset \mathbb{R}$ .

Our main examples still come from Mathematical Finance but now we assume that financial assets can be traded continuously. We assume the same simple situation as in Lecture 1, that is we assume that we have a simple financial market consisting of bonds paying fixed interest, one stock and derivatives of these two assets. We assume that the interest on bonds is compounded continuously, hence the value of the bond at time  $t$  is

$$B_t = B_0 e^{rt}.$$

We denote by  $S_t$  the time  $t$  price of a share and we assume now that  $S_t$  can also change continuously. In order to introduce more precise models for  $S$  similar to discrete time models (like exponential random walk) we need first to define a continuous time analogue to the random walk process which is called *Wiener process* or *Brownian Motion*.

## 4.1 WIENER PROCESS: DEFINITION AND BASIC PROPERTIES

We do not repeat the definitions introduced for discrete time processes in previous lectures if they are exactly the same except for the different time set. Hence we assume that the definitions of the filtration, martingale, ... are already known. On the other hand, some properties of the corresponding objects derived in discrete time can not be taken for granted.

**Definition 4.1** A continuous stochastic process  $\{W_t : t \geq 0\}$  adapted to the filtration  $(\mathcal{F}_t)$  is called an  $(\mathcal{F}_t)$ -Wiener process if

1.  $W_0 = 0$ ,
2. for every  $0 \leq s \leq t$  the random variable  $W_t - W_s$  is independent of  $\mathcal{F}_s$ ,
3. for every  $0 \leq s \leq t$  the random variable  $W_t - W_s$  is normally distributed with mean zero and variance  $(t - s)$ .

In many problems we need to use the so called Wiener process  $(W_t^x)$  started at  $x$ . It is a process defined by the equation

$$W_t^x = x + W_t.$$

The next proposition gives the first consequences of this definition.

**Proposition 4.2** Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process. Then

1.  $EW_t = 0$  and  $EW_s W_t = \min(s, t)$  for all  $s, t \geq 0$  and in particular  $EW_t^2 = t$ .
2. The process  $(W_t)$  is Gaussian.

3. The Wiener process has independent increments, that is, if  $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$  then the random variables  $(W_{t_2} - W_{t_1})$  and  $(W_{t_4} - W_{t_3})$  are independent.

**Proof.** We shall prove (3) first. To this end note that by definition of the Wiener process the random variable  $W_{t_4} - W_{t_3}$  is independent of  $\mathcal{F}_{t_2}$  and  $W_{t_2} - W_{t_1}$  is  $\mathcal{F}_{t_2}$ -measurable. Hence (3) follows from Proposition 2.3 if we put  $\mathcal{F} = \sigma(W_{t_4} - W_{t_3})$  and  $\mathcal{G} = \mathcal{F}_{t_2}$ .

(1) Because  $W_0 = 0$  and  $W_t = W_t - W_0$ , the first part of (1) follows from the definition of Wiener process. Let  $s \leq t$ . Then

$$\begin{aligned} \text{Cov}(W_s, W_t) &= EW_s W_t = EW_s(W_t - W_s) + EW_s^2 \\ &= EW_s E(W_t - W_s) + EW_s^2 = s = \min(s, t). \end{aligned}$$

(2) It is enough to show that for any  $n \geq 1$  and any choice of  $0 \leq t_1 < \dots < t_n$  the random vector

$$X = \begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{pmatrix}$$

is normally distributed. To this end we shall use Proposition 2.9. Note first that by (3) the random variables  $W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$  are independent and therefore the random vector

$$Y = \begin{pmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{pmatrix}$$

is normally distributed. It is also easy to check that  $X = AY$  with the matrix  $A$  given by the equation

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence (2) follows from Proposition 2.9. ■

Note that the properties of Wiener process given in Definition 4.1 are similar to the properties of a random walk process. We shall show that it is not accidental. Let  $S_t(h)$  be the process defined in Exercise 1.8. Let us choose the size of the space grid  $a(h)$  related to the time grid by the condition

$$a^2(h) = h. \tag{4.1}$$

Then using the results of Exercise 1.8 we find immediately that  $ES_t(h) = 0$  and

$$\text{Cov}(S_{t_1}(h), S_{t_2}(h)) = h \left( \frac{\min(t_1, t_2)}{h} \right)$$

Hence

$$\lim_{h \rightarrow 0} \text{Cov}(S_{t_1}(h), S_{t_2}(h)) = \min(t_1, t_2).$$

For every  $t > 0$  and  $h > 0$  the random variable  $S_t(h)$  is a sum of mutually independent and identically distributed random variables. Moreover, if (4.1) holds then all assumption of the

Central Limit Theorem are satisfied and therefore  $S_t(h)$  has an approximately normal distribution for small  $h$  and any  $t = mh$  with  $m$  large. More precisely, for any  $t > 0$ , we find a sequence  $t_n = \frac{kn}{2^n}$  such that  $t_n$  converges to  $t$  and for a fixed  $n$  we choose  $h_n = \frac{1}{2^n}$ . Then we can show that

$$\lim_{n \rightarrow \infty} P(S_{t_n}(h_n) \leq y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right) dz.$$

It can be shown that the limiting process  $(S_t)$  is identical to the Wiener process  $(W_t)$  introduced in Definition 4.1.

**Proposition 4.3** *Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process. Then*

1.  $(W_t)$  is an  $(\mathcal{F}_t)$ -martingale,
2.  $(W_t)$  is a Markov process: for  $0 \leq s \leq t$

$$P(W_t \leq y | \mathcal{F}_s) = P(W_t \leq y | W_s). \quad (4.2)$$

**Proof.** (1) By definition, the process  $(W_t)$  is adapted to the filtration  $(\mathcal{F}_t)$  and  $E|W_t| < \infty$  because it has a normal distribution. The definition of the Wiener process yields also for  $s \leq t$

$$E(W_t - W_s | \mathcal{F}_s) = E(W_t - W_s) = 0$$

which is exactly the martingale property.

(2) We have

$$P(W_t \leq y | \mathcal{F}_s) = E(I_{(-\infty, y]}(W_t) | \mathcal{F}_s) = E(I_{(-\infty, y]}(W_t - W_s + W_s) | \mathcal{F}_s)$$

and putting  $X = W_t - W_s$  and  $Y = W_s$  we obtain (4.2) from Proposition 2.4. ■

The Markov property implies that for every  $t \geq 0$  and  $y \in \mathbb{R}$

$$P(W_t \leq y | W_{s_1} = x_1, W_{s_2} = x_2, \dots, W_{s_n} = x_n) = P(W_t \leq y | W_{s_n} = x_n)$$

for every collection  $0 \leq s_1 \leq \dots \leq s_n \leq t$  and  $x_1, \dots, x_n \in \mathbb{R}$ . On the other hand, the Gaussian property of the Wiener process yields for  $s_n < t$

$$P(W_t \leq y | W_{s_n} = x_n) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi(t-s_n)}} \exp\left(-\frac{(z-x_n)^2}{2(t-s_n)}\right) dz = \Phi\left(\frac{y-x_n}{\sqrt{t-s_n}}\right).$$

Therefore, if we denote by

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right),$$

then  $p(t-s, y-x)$  is the conditional density of  $W_t$  given  $W_s = x$  for  $s < t$ . The density  $p(t-s, y-z)$  is called a transition density of the Wiener process and is sufficient for determining all finite dimensional distributions of the Wiener process. To see this let us consider the random variable  $(W_{t_1}, W_{t_2})$  for  $t_1 < t_2$ . We shall find its joint density. For arbitrary  $x$  and  $y$ , (2.6) yields

$$\begin{aligned} P(W_{t_1} \leq x, W_{t_2} \leq y) &= \int_{-\infty}^x P(W_{t_2} \leq y | W_{t_1} = z) p(t_1, z) dz \\ &= \int_{-\infty}^x \int_{-\infty}^y p(t_2 - t_1, y - z) p(t_1, z) dz \end{aligned}$$

and therefore the joint density of  $(W_{t_1}, W_{t_2})$  is

$$f_{t_1, t_2}(x, y) = p(t_2 - t_1, y - x)p(t_1, x).$$

The same argument can be repeated for the joint distribution of the random vector  $(W_{t_1}, \dots, W_{t_n})$  for any  $n \geq 1$ . The above results can be summarized in the following way. If we know the initial position of the Markov process at time  $t = 0$  and we know the transition density, then we know all finite dimensional distributions of the process. This means that in principle we can calculate any functional of the process of the form  $E[F(W_{t_1}, \dots, W_{t_n})]$ . In particular, for  $n = 2$ ,  $E[F(W_{t_1}, W_{t_2})] = \int_{-\infty}^x \int_{-\infty}^y F(x, y)p(t_2 - t_1, y - z)p(t_1, z)dz$ , for  $t_1 < t_2$ .

The next proposition shows the so called *invariance properties* of the Wiener process.

**Proposition 4.4** *Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process.*

1. *For fixed  $s \geq 0$ , we define a new process  $V_t = W_{t+s} - W_s$ . Then the process  $(V_t)$  is also a Wiener process with respect to the filtration  $(\mathcal{G}_t) = (\mathcal{F}_{t+s})$ .*
2. *For a fixed  $c > 0$  we define a new process  $U_t = cW_{t/c^2}$ . Then the process  $(U_t)$  is also a Wiener process with respect to the filtration  $(\mathcal{H}_t) = (\mathcal{F}_{t/c^2})$ .*

**Proof.** (1) Clearly  $V_0 = 0$  and the random variable  $V_t$  is  $\mathcal{F}_{t+s} = \mathcal{G}$ -measurable. For any  $h > 0$  we have  $V_{t+h} - V_t = W_{t+h+s} - W_{t+s}$  and therefore  $V_{t+h} - V_t$  is independent of  $\mathcal{F}_{t+h} = \mathcal{G}_t$  with Gaussian  $N(0, h)$  distribution. Hence  $V$  is a Wiener process.

(2) The proof of (2) is similar and left as an exercise. ■

We shall apply the above results to calculate some quantities related to simple transformations of the Wiener process.

**Example 4.1** *Let  $T > 0$  be fixed and let*

$$B_t = W_T - W_{T-t}.$$

*Then the process  $B$  is a Wiener process on the time interval  $[0, T]$  with respect to the filtration  $\mathcal{G}_t = \sigma(W_T - W_{T-s} : s \leq t)$ .*

**Proof.** *Note first that the process  $B$  is adapted to the filtration  $(\mathcal{G}_t)$ . We have also for  $0 \leq s < t \leq T$*

$$B_t - B_s = W_{T-s} - W_{T-t}$$

*and because the Wiener process  $W$  has independent increments, we can check easily that  $B_t - B_s$  is independent of  $\mathcal{G}_s$ . Obviously  $B_t - B_s$  has the Gaussian  $N(0, t - s)$  distribution and  $B_0 = 0$ . Because  $B$  is a continuous process, all conditions of Definition 4.1 are satisfied and  $B$  is a Wiener process. ■*

**Example 4.2** *Let*

$$S_t = mt + \sigma W_t.$$

*The process  $(S_t)$  is called a Wiener process with drift  $m$  and variance  $\sigma^2$ . Clearly  $ES_t = mt$  and  $E(S_t - mt)^2 = \sigma^2 t$ . We shall determine the joint density of the random variables  $(W_{t_1}, S_{t_2})$ . Note that  $S$  is a Gaussian process and the random variable  $(W_{t_1}, S_{t_2}) = (W_{t_1}, mt_2 + \sigma W_{t_2})$  is jointly Gaussian as well (can you show that?). By the definition of the Wiener process we obtain*

$$\text{Cov}(W_{t_1}, mt_2 + \sigma W_{t_2}) = \sigma EW_{t_1}W_{t_2} = \sigma \min(t_1, t_2).$$

Hence the covariance matrix is

$$C = \begin{pmatrix} t_1 & \sigma \min(t_1, t_2) \\ \sigma \min(t_1, t_2) & \sigma^2 t_2 \end{pmatrix}$$

and

$$(W_{t_1}, S_{t_2}) \sim N \left( \begin{pmatrix} 0 \\ mt_2 \end{pmatrix}, \begin{pmatrix} t_1 & \sigma \min(t_1, t_2) \\ \sigma \min(t_1, t_2) & \sigma^2 t_2 \end{pmatrix} \right).$$

**Example 4.3** Let

$$Y_t = xe^{mt + \sigma W_t}.$$

The process  $Y$  is called an exponential Wiener process and ubiquitous in mathematical finance. This process is a continuous time analogue of the exponential random walk. Note first that by Theorem 2.2, the process  $Y$  is adapted to the filtration  $(\mathcal{F}_t)$ . By (2.7)

$$EY_t = xe^{(m + \frac{1}{2}\sigma^2)t}.$$

We shall show now that the process  $Y$  is an  $(\mathcal{F}_t)$ -martingale if and only if

$$m = -\frac{\sigma^2}{2}.$$

Indeed, for  $s \leq t$  we obtain

$$\begin{aligned} E(xe^{mt + \sigma W_t} | \mathcal{F}_s) &= E(xe^{ms + m(t-s) + \sigma W_s} e^{\sigma(W_t - W_s)} | \mathcal{F}_s) \\ &= Y_s E e^{m(t-s) + \sigma(W_t - W_s)} \\ &= Y_s e^{m(t-s)} E e^{\sigma W_{t-s}} \\ &= Y_s e^{m(t-s) + \sigma^2(W_t - W_s)} \end{aligned}$$

and the claim follows.

In the next example we shall need the following:

**Lemma 4.5** Let  $X$  be a stochastic process such that

$$\int_0^T E |X_s| ds < \infty.$$

Then

$$E \int_0^T X_s ds = \int_0^T EX_s ds.$$

Moreover, if the process  $X$  is adapted to the filtration  $(\mathcal{F}_t)$ , then the process

$$Y_t = \int_0^t X_s ds, \quad t \leq T$$

is also adapted to  $(\mathcal{F}_t)$ . Additionally, if  $X$  is a Gaussian process, then the process  $Y$  is also Gaussian.

**Proof.** We give only a very brief idea of the proof. By assumption the process  $X$  is integrable on  $[0, T]$  and hence we can define an approximating sequence

$$Y_t^n = \sum_{0 < kT/2^n \leq t} X_{\frac{k-1}{2^n}T} \left( \frac{kT}{2^n} - \frac{(k-1)T}{2^n} \right)$$

for the process  $(Y_t)$ . It is not difficult to see that all of the properties stated in the lemma hold if we replace  $Y$  with  $Y^n$ . By definition of the integral

$$\lim_{n \rightarrow \infty} Y_t^n = Y_t$$

for every  $t > 0$  and it remains to prove only that the properties stated in the lemma are preserved in the limit. This part of the proof is omitted. ■

**Example 4.4** Consider the process

$$Y_t = \int_0^t W_s ds.$$

By Lemma 4.5 it is an  $(\mathcal{F}_t)$ -adapted Gaussian process. We shall determine the distribution of the random variable  $Y_t$  for a fixed  $t$ . By Lemma 4.5 we have  $EY_t = 0$ . Now, for  $s \leq t$ , once more by Lemma 4.5

$$\begin{aligned} EY_s Y_t &= E \left( \int_0^s W_u du \right) \left( \int_0^t W_v dv \right) = \int_0^s \int_0^t EW_u W_v dudv \\ &= \int_0^s \int_0^s EW_u W_v dudv + \int_0^s \int_s^t EW_u W_v dudv \\ &= \int_0^s \int_0^s \min(u, v) dudv + \int_0^s \int_s^t EW_u (W_v - W_s) dudv + \int_0^s \int_s^t EW_u W_s dudv \\ &= \frac{1}{3}s^3 + \int_0^s (t-s)udu = \frac{1}{3}s^3 + \frac{1}{2}(t-s)s^2 \end{aligned}$$

because

$$\int_0^s \int_s^t EW_u (W_v - W_s) dudv = 0.$$

Hence

$$EY_s Y_t = \frac{1}{3} \min(s^3, t^3) + \frac{1}{2}(t-s) \min(s^2, t^2).$$

## 4.2 WIENER PROCESS: PROPERTIES OF SAMPLE PATHS

We did not specify the sample space of the Wiener process yet. In analogy with the discussion in Lecture 1 we identify the sample point with the whole *Brownian path*

$$\omega = \text{trajectory } t \rightarrow W_t(\omega)$$

for  $t \geq 0$  or  $t \leq T$ , which is a continuous function of time by definition. We shall show that Brownian paths are extremely irregular. We start with some introductory remarks.

Consider a function  $f : [0, \infty) \rightarrow \mathbb{R}$ , such that  $f(0) = 0$  and

$$\int_0^T |f'(t)|^2 dt < \infty,$$

in which case

$$f(t) = \int_0^t f'(s) ds.$$

Let us calculate how fast are the oscillations of the function  $f$  on a fixed interval  $[0, T]$ . To this end for every  $n \geq 1$  we divide  $[0, T]$  into a sequence of  $k_n$  subintervals

$$P_1^n = (0, t_1^n), P_2^n = (t_1^n, t_2^n), \dots, P_{k_n}^n = (t_{k_n-1}^n, T).$$

The whole division will be denoted by  $P^n$ . With every division  $P^n$  of  $[0, T]$  we associate its "size" defined by the equation

$$d(P^n) = \max_{i \leq k_n} (t_{k_i}^n - t_{k_{i-1}}^n),$$

where  $t_0^n = 0$  and  $t_{k_n}^n = T$ . For a given division  $P^n$  we define the corresponding measure of the oscillation of the function  $f$

$$V_n(f) = \sum_{i=1}^{k_n} \left| f(t_{k_i}^n) - f(t_{k_{i-1}}^n) \right|.$$

Then

$$V_n(f) = \sum_{i=1}^{k_n} \left| \int_{t_{k_{i-1}}^n}^{t_{k_i}^n} f'(y) dy \right| \leq \sum_{i=1}^{k_n} \int_{t_{k_{i-1}}^n}^{t_{k_i}^n} |f'(y)| dy = \int_0^T |f'(y)| dy.$$

It follows that

$$V_n(f) \leq \int_0^T |f'(y)| dy < \infty$$

and this bound is independent of the choice of the division  $P^n$ . If

$$\sup_{n \geq 1} V_n(f) < \infty$$

then the limit

$$V(f) = \lim_{d(P^n) \rightarrow \infty} V_n(f)$$

exists and is called a variation of the function  $f$ . A function with this property is called a *function of bounded variation*.

For a given division  $P^n$  we shall calculate now the so-called *quadratic variation*

$$V_n^{(2)}(f) = \sum_{i=1}^{k_n} |f(t_{k_i}^n) - f(t_{k_{i-1}}^n)|^2$$

of the function  $f$ . Using the same arguments as above we obtain by Schwartz inequality

$$\begin{aligned} V_n^{(2)}(f) &\leq \sum_{k=1}^{k_n} \left( \int_{t_{k_{i-1}}^n}^{t_{k_i}^n} |f'(y)| dy \right)^2 \leq \sum_{k=1}^{k_n} (t_{k_i}^n - t_{k_{i-1}}^n) \int_{t_{k_{i-1}}^n}^{t_{k_i}^n} |f'(y)|^2 dy \\ &\leq \max_{i < k_n} (t_{k_i}^n - t_{k_{i-1}}^n) \sum_{k=1}^{k_n} \int_{t_{k_{i-1}}^n}^{t_{k_i}^n} |f'(y)|^2 dy = \max_{i \leq k_n} (t_{k_i}^n - t_{k_{i-1}}^n) \int_0^T |f'(y)|^2 dy. \end{aligned}$$

Finally

$$V_n^{(2)}(f) \leq d(P^n) \int_0^T |f'(y)|^2 dy \tag{4.3}$$

and therefore

$$\lim_{d(P^n) \rightarrow 0} V_n^{(2)}(f) = 0.$$

We say that the function  $f$  with the square integrable derivative has *zero quadratic variation*.

We shall show that the behavior of brownian paths is very different from the behavior of differentiable functions described above.

**Definition 4.6** A stochastic process  $\{X_t : t \geq 0\}$  is of finite quadratic variation if there exists a process  $\langle X \rangle$  such that for every  $t \geq 0$

$$\lim_{d(P^n) \rightarrow 0} E (V_n^{(2)}(X) - \langle X \rangle_t)^2 = 0,$$

where  $P^n$  denotes a division of the interval  $[0, t]$ .

**Theorem 4.7** A Wiener process is of finite quadratic variation and  $\langle W \rangle_t = t$  for every  $t \geq 0$ .

**Proof.** For a fixed  $t > 0$  we need to consider

$$V_n^{(2)}(W) = \sum_{i=1}^{k_n} \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2.$$

Note first that

$$E V_n^{(2)}(W) = \sum_{i=1}^{k_n} E \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2 = \sum_{i=1}^{k_n} (t_i^n - t_{i-1}^n) = t$$

and therefore

$$E (V_n^{(2)}(W) - t)^2 = E \left( \sum_{i=1}^{k_n} \left( \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2 - (t_i^n - t_{i-1}^n) \right) \right)^2.$$

Because increments of the Wiener process are independent

$$E (V_n^{(2)}(W) - t)^2 = \sum_{i=1}^{k_n} E \left( \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2 - (t_i^n - t_{i-1}^n) \right)^2.$$

Finally, we obtain

$$\begin{aligned} E (V_n^{(2)}(W) - t)^2 &\leq 2 \sum_{i=1}^{k_n} (t_i^n - t_{i-1}^n)^2 \leq \sup_{i \leq k_n} 2 (t_i^n - t_{i-1}^n) \sum_{i=1}^{k_n} (t_i^n - t_{i-1}^n) \\ &= 2td(P^n) \end{aligned}$$

and therefore  $\langle W \rangle_t = t$  as desired. ■

**Corollary 4.8** Brownian paths are of infinite variation on any interval:

$$P (V(W) = \infty) = 1.$$

**Proof.** Let  $[0, t]$  be such an interval that for a certain sequence  $P^n$  of division

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2 = t.$$

Then

$$\sum_{i=1}^{k_n} \left( W_{t_i^n} - W_{t_{i-1}^n} \right)^2 \leq \sup_{i \leq k_n} \left| W_{t_i^n} - W_{t_{i-1}^n} \right| \sum_{i=1}^{k_n} \left| W_{t_i^n} - W_{t_{i-1}^n} \right|.$$

Now the left hand side of this inequality tends to  $t$  and continuity of the Wiener process yields

$$\lim_{n \rightarrow \infty} \sup_{i \leq k_n} \left| W_{t_i^n} - W_{t_{i-1}^n} \right| = 0.$$

Therefore necessarily

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \left| W_{t_i^n} - W_{t_{i-1}^n} \right| = \infty.$$

■

**Theorem 4.9** *Brownian paths are nowhere differentiable.*

Consider the process  $M_t = W_t^2 - t$  which is obviously adapted. We have also

$$\begin{aligned} E(M_t | \mathcal{F}_s) &= E(W_t^2 - t | \mathcal{F}_s) = E((W_t - W_s)^2 + 2W_s W_t - W_s^2 - t | \mathcal{F}_s) \\ &= E((W_t - W_s)^2 | \mathcal{F}_s) + 2W_s E(W_t | \mathcal{F}_s) - W_s^2 - t \\ &= (t - s) + 2W_s^2 - W_s^2 - t = M_s \end{aligned}$$

and therefore the process  $(M_t)$  is an  $(\mathcal{F}_t)$ -martingale. We can rephrase this result by saying that the process  $M_t = W_t^2 - \langle W \rangle_t$  is a martingale, the property which will be important in the future.

### 4.3 EXERCISES

1.

- (a) Let  $\{Y_t : t \geq 0\}$  be a Gaussian stochastic process and let  $f, g : [0, \infty) \rightarrow \mathbb{R}$  be two functions. Show that the process  $X_t = f(t) + g(t)Y_t$  is Gaussian. Deduce that the process  $X_t = x + at + \sigma W_t$ , where  $W$  is a Wiener process is Gaussian for any choice of  $a \in \mathbb{R}$  and  $\sigma > 0$ .
- (b) Show that  $(X_t)$  is a Markov process. Find  $EX_t$  and  $\text{Cov}(X_s, X_t)$ .
- (c) Find the joint distribution of  $(X_{t_1}, X_{t_2})$ . Compute

$$E(e^{X_{t_2}} | X_{t_1})$$

for  $t_1 \leq t_2$ .

- (d) Find the distribution of the random variable

$$Z_t = \int_0^t X_s ds.$$

- (e) Compute  $\langle X \rangle_t$ .
- (f) Show that the process  $X_t = -W_t$  is also a Wiener process.

2. (!) Let  $u(t, x) = Ef(x + W_t)$  for a certain bounded function  $f$ . Use the definition of expected value to show that

$$u(t, x) = \int_{-\infty}^{\infty} f(x + y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) dy.$$

Using change of variables show that the function  $u$  is twice differentiable in  $x$  and differentiable in  $t > 0$ . Finally, show that

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x).$$

3. (!) Let  $S$  be an exponential Wiener process starting from  $S_0 = x > 0$ .
- Find the density of the random variable  $S_t$  for  $t > 0$ .
  - Find the mean and variance of  $S_t$ .
  - Let for  $x > 0$   $u(t, x) = Ef(xS_t)$  for a bounded function  $f$ . Show that  $u$  has the same differentiability properties as in (2) and

$$\frac{\partial u}{\partial t}(t, x) = \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + (m + \frac{1}{2}\sigma^2)x \frac{\partial u}{\partial x}(t, x).$$

4. Let  $(W_t)$  be a Wiener process and let  $(\mathcal{F}_t)$  denote its natural filtration.
- Compute the covariance matrix  $C$  for  $(W_s, W_t)$ .
  - Use the covariance matrix  $C$  to write down the joint density  $f(x, y)$  for  $(W_s, W_t)$ .
  - Change the variable in the density  $f$  to compute the joint density  $g$  for the pair of random variables  $(W_s, W_t - W_s)$ . Note this density factors into a density for  $W_s$  and a density for  $W_t - W_s$ , which shows that these two random variables are independent.
  - Use the density  $g$  in (c) to compute

$$\phi(u, v) = E \exp(uW_s + vW_t).$$

Verify that

$$\phi(u, v) = \exp\left(\frac{1}{2}(u, v)C \begin{pmatrix} u \\ v \end{pmatrix}\right).$$

5. Let  $\{X_t : t \geq 0\}$  be a continuous and a Gaussian process, such that  $EX_t = 0$  and  $E(X_s X_t) = \min(s, t)$ . Show that the process  $(X_t)$  is a Wiener process with respect to the filtration  $(\mathcal{F}_t^X)$ .
6. (**Brownian bridge**) Let  $(W_t)$  be a Wiener process. Let  $x, y$  be arbitrary real numbers. The Brownian bridge between  $x$  and  $y$  is defined by the formula

$$V_t^{x,y} = x \left(1 - \frac{t}{T}\right) + W_t - \frac{t}{T} (W_T - y), \quad t \leq T.$$

- (a) Show that

$$P(W_t \leq a \mid W_T = y) = P(V_t^{0,y} \leq a), \quad t \leq T.$$

Deduce that for all  $t \leq T$

$$P(W_t \leq a) = \int_{-\infty}^{\infty} P(V_t^{0,y} \leq a) f_T(y),$$

where  $f_T$  denotes the density of  $W_T$ .

(b) For any  $t \leq T$  and any  $0 \leq t_1 \leq \dots \leq t_n \leq T$  show that

$$P(W_{t_1} = a_1, \dots, W_{t_n} \leq a_n \mid W_T = y) = P(V_{t_1}^{0,y} \leq a_1, \dots, V_{t_n}^{0,y} \leq a_n)$$

for any real  $a_1, \dots, a_n$ .

(c) Compute  $EV_t^{x,y}$  and  $\text{Cov}(V_{t_1}^{x,y}, V_{t_2}^{x,y})$  for  $t_1, t_2 \leq T$  and arbitrary  $x, y$ .

(d) Show that the process  $\{V_{T-t}^{x,y} : t \leq T\}$  is also a Brownian bridge process.

7. Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process and let  $X_t = |W_t|$ .

(a) Find  $P(X_t \leq x)$  and evaluate the density of the random variable  $X_t$ .

(b) Write down the formula for  $P(X_s \leq x, X_t \leq y)$  as a double integral from a certain function and derive the joint density of  $(X_s, X_t)$ . Consider all possible values of  $s, t$  and  $x, y$ .

(c) For  $s < t$  find  $P(X_t \leq y \mid X_s = x)$  in the integral form and compute the conditional density of  $X_t$  given  $X_s = x$ .

(d) Is  $(X_t)$  a Markov process?

# Chapter 5 STRONG MARKOV PROPERTY AND REFLECTION PRINCIPLE

A good reference for this chapter is:

*Karatzas I. and Shreve S. E.: Stochastic Calculus and Brownian Motion*

Let  $(W_t)$  be a real-valued  $(\mathcal{F}_t)$ -Wiener process. For  $b > 0$  we define a random variable

$$\tau_b = \min \{t \geq 0 : W_t = b\}. \quad (5.1)$$

It is clear that

$$\{\tau_b \leq t\} = \left\{ \max_{s \leq t} W_s \geq b \right\}.$$

This identity shows that the event  $\{\tau_b \leq t\}$  is completely determined by the past of the Wiener process up to time  $t$  and therefore for every  $t \geq 0$

$$\{\tau_b \leq t\} \in \mathcal{F}_t$$

that is,  $\tau_b$  is a stopping time. We shall compute now its probability distribution. Note first that

$$P(\tau_b < t) = P(\tau_b < t, W_t > b) + P(\tau_b < t, W_t < b) = P(W_t > b) + P(\tau_b < t, W_t < b).$$

Using in a heuristic way the identity

$$P(\tau_b < t, W_t < b) = P(\tau_b < t, W_t > b) = P(W_t > b)$$

we obtain

$$\begin{aligned} P(\tau_b < t) &= 2P(W_t > b) = 2 \int_{b/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= 2 \left( 1 - \Phi \left( \frac{b}{\sqrt{t}} \right) \right). \end{aligned} \quad (5.2)$$

Hence for every  $b > 0$  the stopping time  $\tau_b$  has the density

$$f_b(t) = \frac{b}{\sqrt{2\pi t^3}} \exp \left( -\frac{b^2}{2t} \right), t \geq 0. \quad (5.3)$$

Let us recall the result Proposition 4.3 from Lecture 4 which says that the Wiener process starts afresh at any moment of time  $s \geq 0$  in the sense that the process  $B_t = W_{s+t} - W_s$  is also a Wiener process. The intuitive argument applied to derive (5.2) is based on the assumption that the Wiener process starts afresh if the fixed moment of time  $s$  is replaced with a stopping time  $\tau_b$ , that is, it is implicitly assumed that the process  $W_{\tau_b+t} - W_{\tau_b}$  is also a Wiener process. The rigorous argument is based on the following *strong Markov property*:

$$P(W_{\tau+t} \leq x \mid \mathcal{F}_\tau) = P(W_{\tau+t} \leq x \mid W_\tau) \quad (5.4)$$

for all  $x \in \mathbb{R}$ . In this definition we assume that  $\tau$  is a finite stopping time, that is  $P(\tau < \infty) = 1$ . In the same way as in the case of the Markov property (5.4) yields

$$E(f(W_{\tau+t}) \mid \mathcal{F}_\tau) = E(f(W_{\tau+t}) \mid W_\tau).$$

**Theorem 5.1** *The Wiener process  $(W_t)$  enjoys the strong Markov property. Moreover, if  $\tau$  is any stopping time then the process  $B_t = W_{\tau+t} - W_\tau$  is a Wiener process with respect to the filtration  $(\mathcal{G}_t) = (\mathcal{F}_{\tau+t})$  and the process  $(B_t)$  is independent of  $\mathcal{G}_0$ .*

**Corollary 5.2** *Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process and for  $b > 0$  let  $\tau_b$  be given by (5.1). Then the density of  $\tau_b$  is given by (5.3).*

It follows from (5.2) that

$$P(\tau_b < \infty) = \lim_{t \rightarrow \infty} P(\tau_b \leq t) = \lim_{t \rightarrow \infty} 2 \left( 1 - \Phi \left( \frac{b}{\sqrt{t}} \right) \right) = 1.$$

It means that the Wiener process cannot stay forever below any fixed level  $b$ .

Theorem 5.1 allows us to calculate many important probabilities related to the Wiener process. Note first that for the random variable

$$W_t^* = \max_{s \leq t} W_s$$

we obtain for  $x \geq 0$

$$P(W_t^* \leq x) = P(\tau_x > t) = 2\Phi \left( \frac{x}{\sqrt{t}} \right) - 1 = P(|W_t| \leq x).$$

Hence the random variables  $W_t^*$  and  $|W_t|$  have the same distributions.

**Proposition 5.3** *For  $t > 0$  the random variable  $(W_t, W_t^*)$  has the density*

$$f(x, y) = \begin{cases} \frac{2(2y-x)}{t} \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(2y-x)^2}{2t} \right) & \text{if } x \leq y, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** *Note first that by the symmetry of Wiener process*

$$P(b + W_s \leq a) = P(W_s \geq b - a) = P(b + W_s \geq 2b - a).$$

Hence denoting  $B_t = W_{\tau_y+t} - W_{\tau_y}$  and using the strong Markov property of the Wiener process we obtain

$$\begin{aligned} P(W_t \leq x, W_t^* \geq y) &= P(W_t \leq x \mid \tau_y \leq t) P(\tau_y \leq t) \\ &= P(y + W_t - W_{\tau_y} \leq x \mid \tau_y \leq t) P(\tau_y \leq t) \\ &= P(y + B_{t-\tau_y} \leq x \mid \tau_y \leq t) P(\tau_y \leq t) \\ &= P(y + B_{t-\tau_y} \geq 2y - x \mid \tau_y \leq t) P(\tau_y \leq t) \\ &= P(y + W_t - W_{\tau_y} \geq 2y - x \mid \tau_y \leq t) P(\tau_y \leq t) \\ &= P(W_t \geq 2y - x \mid W_t^* \geq y) P(W_t^* \geq y) \\ &= P(W_t \geq 2y - x, W_t^* \geq y) \\ &= P(W_t \geq 2y - x) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{2y-x}^{\infty} e^{-z^2/2t} dz. \end{aligned}$$

Hence

$$P(W_t \leq x, W_t^* \leq y) = P(W_t \leq x) - P(W_t \leq x, W_t^* \geq y)$$

and because

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} P(W_t \leq x, W_t^* \leq y)$$

the proposition follows easily. ■

Note that by the invariance property of the Wiener process

$$P\left(\max_{s \leq t} W_s \geq b\right) = P\left(\max_{s \leq t} (-W_s) \geq b\right) = P\left(\min_{s \leq t} W_s \leq -b\right).$$

Proposition 5.3 allows us to determine the distribution of the random variable  $W_t^* - W_t$ . We start with simple observation that

$$W_t^* - W_t = \max_{s \leq t} (W_s - W_t) = \max_{t-s \geq 0} (W_{t-s} - W_t).$$

Since the process  $B_s = W_t - W_{t-s}$  is a Wiener process on the time interval  $[0, t]$  we find that

$$P(W_t^* - W_t \leq x) = P\left(\max_{s \leq t} B_s \leq x\right) = P(|B_t| \leq x).$$

Note that we have proved that the random variables  $|W_t|$ ,  $W_t^*$ , and  $W_t^* - W_t$  have the same density (5.3).

We end up with one more example of calculations related to the distribution of the maximum of the Wiener process. Let  $X_t = x + W_t$  with  $x > 0$ . We shall calculate

$$P\left(X_t \leq y \mid \min_{s \leq t} X_s > 0\right).$$

For  $y \leq 0$  this probability is equal to zero. Let  $y > 0$ . By the definition of the process  $X$

$$\begin{aligned} P\left(X_t \leq y \mid \min_{s \leq t} X_s > 0\right) &= P\left(W_t \leq y - x \mid \min_{s \leq t} W_s > -x\right) \\ &= P\left(-W_t \geq x - y \mid -\min_{s \leq t} W_s < x\right) \\ &= P\left(-W_t \geq x - y \mid \max_{s \leq t} (-W_s) < x\right) \\ &= P\left(B_t \geq x - y \mid \max_{s \leq t} B_s < x\right) \\ &= \frac{P(B_t \geq x - y, \max_{s \leq t} B_s < x)}{P(\max_{s \leq t} B_s < x)} \end{aligned}$$

where  $B_t = -W_t$  is a new Wiener process. Now, we have

$$P\left(B_t \geq x - y, \max_{s \leq t} B_s < x\right) = P(B_t \geq x - y) - P(B_t \geq x - y, B_t^* \geq x)$$

and by Proposition (5.3)

$$P(B_t \geq x - y, B_t^* \geq x) = \int_{x-y}^{\infty} \int_x^{\infty} \frac{2(2v-u)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2v-u)^2}{2t}\right) dv du.$$

Note that

$$\begin{aligned} \frac{\partial}{\partial y} P(B_t \geq x - y, B_t^* \geq x) &= - \int_x^{\infty} \frac{2(2v - (x - y))}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2v - (x - y))^2}{2t}\right) dv \\ &= \int_x^{\infty} \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(2v - (x - y))^2}{2t}\right) \right) dv \\ &= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x + y)^2}{2t}\right). \end{aligned}$$

Therefore the conditional density of  $X_t$  given that  $\min_{s \leq t} X_s > 0$  is

$$\begin{aligned} f\left(y \mid \min_{s \leq t} X_s > 0\right) &= \frac{\partial}{\partial y} P\left(X_t \leq y \mid \min_{s \leq t} X_s > 0\right) \\ &= \frac{1}{P(\tau_x \leq t)} \left( \frac{\partial}{\partial y} P(B_t \geq x - y) - \frac{\partial}{\partial y} P(B_t \geq x - y, B_t^* \geq x) \right) \\ &= \frac{1}{P(\tau_x \leq t)} \left( \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x+y)^2}{2t}\right) \right). \end{aligned}$$

We end up this lecture with some properties of the distribution of the stopping time  $\tau_b$ . If  $(W_t)$  is a Wiener process then the process  $W_t^{(b)} = b^{-1}W_{b^2t}$  is also a Wiener process. We define a stopping time

$$\tau_1^{(b)} = \min \left\{ t \geq 0 : W_t^{(b)} = 1 \right\}.$$

**Proposition 5.4** *We have  $\tau_b = b^2\tau_1^{(b)}$  and consequently  $\tau_b$  has the same density as the random variable  $b^2\tau_1^{(b)}$ . Moreover, the random variables  $\tau_b$ ,  $\left(\frac{b}{W_1^*}\right)^2$  and  $\left(\frac{b}{W_1}\right)^2$  have the same densities.*

**Proof.** *By definition*

$$\begin{aligned} \tau_b &= \min \left\{ t \geq 0 : b^{-1}W_t = 1 \right\} = \min \left\{ b^2t \geq 0 : W_t^{(b)} = 1 \right\} \\ &= b^2 \min \left\{ t \geq 0 : W_t^{(b)} = 1 \right\} = b^2\tau_1^{(b)}. \end{aligned}$$

*In order to prove the remaining properties it is enough to put  $b = 1$  and show that  $\tau_1$  has the same density as  $\frac{1}{(W_1^*)^2}$ . Note first that  $W_t^*$  has the same density as  $\sqrt{t}W_1^*$ . Therefore*

$$P(\tau_1 \geq x) = P(W_x^* \leq 1) = P(\sqrt{x}W_1^* \leq 1) = P\left(\frac{1}{(W_1^*)^2} \geq x\right)$$

*and this concludes the proof. ■*

## 5.1 EXERCISES

1. Let  $X_t = x + \sigma W_t$ , where  $(W_t)$  is a Wiener process and  $\sigma, x > 0$  are fixed. Let

$$\tau_0 = \min \{t \geq 0 : X_t = 0\}.$$

- Find the conditional density of  $X_t$  given  $X_s = x$  for  $s \leq t$ .
- Find the density of the stopping time  $\tau_0$ .
- Compute  $P(X_t \leq y, \tau_0 > t)$ .
- Let

$$Y_t = \begin{cases} X_t & \text{if } t \leq t_0, \\ 0 & \text{if } t > t_0. \end{cases}$$

Find  $P(Y_t \leq y \mid Y_s = x)$  for  $s \leq t$  and the conditional density of  $Y_t$  given  $X_s = x$ .

2. Let  $\tau_b$  be the stopping time (5.1). Show that  $E\tau_b = \infty$ .

3. For a fixed  $t_0 > 0$  we define the barrier

$$b(t) = \begin{cases} a & \text{if } t \leq t_0, \\ b & \text{if } t > t_0, \end{cases}$$

where  $0 < a < b$ . Let  $W$  be a Wiener process and

$$\tau = \min \{t \geq 0 : W_t \geq b(t)\}.$$

Compute  $P(\tau \leq t)$  for  $t \geq 0$ .

4. Let  $\tau_b$  be defined by (5.1). Using the Strong Markov Property show that for  $0 \leq a < b$

$$\tau_b - \tau_a = \inf \{t \geq 0 : W_{\tau_a+t} - W_{\tau_a} = b - a\}.$$

Derive that the random variable  $\tau_b - \tau_a$  is independent of the  $\sigma$ -algebra  $\mathcal{F}_{\tau_a}^W$ . Finally, show that the stochastic process  $\{\tau_a : a \geq 0\}$  has independent increments.

5. Let  $W$  be a Wiener process. Show that the conditional density of the pair  $(W_{t+s}, W_{t+s}^*)$  given  $W_t = a$  and  $W_t^* = b$  is

$$f(x, y | a, b) = \frac{2(2y - x - a)}{\sqrt{2\pi s^3}} \exp\left(-\frac{(2y - x - a)^2}{2s}\right).$$

6. Find the joint distribution of the random variable  $(W_s, W_t^*)$  for  $s \neq t$ .

7. Let  $W_t$  and  $B_t$  be two independent Wiener processes.

(a) Show that the random variables

$$\frac{B_1}{W_1} \quad \text{and} \quad \frac{B_1}{|W_1|}$$

have the same *Cauchy density*

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

(b) Let  $\tau_b$  be the stopping time defined by (5.1) for the process  $(W_t)$ . Apply Proposition 5.4 to show that the random variable  $B_{\tau_b}$  has the same distribution as the random variable  $\frac{b}{|W_1|}B_1$  and deduce from (a) the density of the random variable  $B_{\tau_b}$ .

# Chapter 6 MULTIDIMENSIONAL WIENER PROCESS

Reference: Karatzas I. and Shreve S. E.: *Stochastic Calculus and Brownian Motion*

Let  $W^1, W^2, \dots, W^d$  be a family of  $d$  independent Wiener processes adapted to the same filtration  $(\mathcal{F}_t)$ . An  $\mathbb{R}^d$ -valued process

$$W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^d \end{pmatrix}$$

is called a  $d$ -dimensional Wiener process. For this process the following properties hold.

**Proposition 6.1** *Let  $(W_t)$  be a  $d$ -dimensional Wiener process adapted to the filtration  $(\mathcal{F}_t)$ . Then*

1.  $EW_t = 0$  and  $EW_s W_t^T = \min(s, t)I$  for all  $s, t \geq 0$  and in particular  $EW_t W_t^T = tI$ .
2. The process  $(W_t)$  is Gaussian.
3. The Wiener process  $(W_t)$  has independent increments.

**Proposition 6.2** *Let  $(W_t)$  be a  $d$ -dimensional Wiener process adapted to the filtration  $(\mathcal{F}_t)$ . Then*

1.  $(W_t)$  is an  $(\mathcal{F}_t)$ -martingale.
2.  $(W_t)$  is a Markov process: for  $0 \leq s \leq t$  and real numbers  $y_1, \dots, y_d$

$$P(W_t^1 \leq y_1, \dots, W_t^d \leq y_d \mid \mathcal{F}_s) = P(W_t^1 \leq y_1, \dots, W_t^d \leq y_d \mid W_s). \quad (6.1)$$

Let  $(X_t)$  and  $(Y_t)$  be two real-valued continuous stochastic processes. For any division  $(P^n)$  of the interval  $[0, t]$  we define

$$V_n^{(2)}(X, Y) = \sum_{l=1}^{k_n} \left( X_{t_l^n} - X_{t_{l-1}^n} \right) \left( Y_{t_l^n}^j - Y_{t_{l-1}^n}^j \right). \quad (6.2)$$

**Definition 6.3** *Let  $X$  and  $Y$  be two  $\mathbb{R}$ -valued continuous processes. We say that  $X$  and  $Y$  have joint quadratic variation process  $\langle X, Y \rangle$  if for every  $t \geq 0$  there exists random variables  $\langle X, Y \rangle_t$  such that*

$$\lim_{d(P^n) \rightarrow 0} E \left( V_n^{(2)}(X, Y) - \langle X, Y \rangle_t \right)^2 = 0.$$

If  $X = Y$  then we write  $\langle X \rangle$  instead of  $\langle X, X \rangle$  and call it quadratic variation process of  $X$ . If  $(X_t)$  and  $(Y_t)$  are two  $\mathbb{R}^d$ -valued continuous stochastic processes, that is  $X_t^T = (X_t^1, \dots, X_t^d)$  and  $Y_t^T = (Y_t^1, \dots, Y_t^d)$  such that  $\langle X^i, Y^j \rangle_t$  exists for all  $i, j = 1, \dots, d$  then the matrix-valued process  $\langle X, Y \rangle_t = (\langle X^i, Y^j \rangle)_{i, j \leq d}$  is called a joint quadratic variation of the processes  $(X_t)$  and  $(Y_t)$ .

Hence for vector-valued processes quadratic variation is a matrix valued stochastic process which can be defined using matrix notation

$$\langle X, Y \rangle_t = \lim_{d(P^n) \rightarrow 0} \sum_{l=1}^{k_n} (X_{t_l} - X_{t_{l-1}}) (Y_{t_l} - Y_{t_{l-1}})^T, \quad (6.3)$$

where  $P^n, n \geq 1$ , is a sequence of divisions of  $[0, t]$ .

**Lemma 6.4** *If  $X, Y, Z$  are real valued processes of finite quadratic variation then*

$$\langle X, Y \rangle_t = \langle Y, X \rangle_t = \frac{1}{4} \langle X + Y \rangle_t - \frac{1}{4} \langle X - Y \rangle_t \quad (6.4)$$

and

$$\langle aX + bY, Z \rangle_t = a \langle X, Z \rangle_t + b \langle Y, Z \rangle_t. \quad (6.5)$$

**Proof.** *The symmetry of joint quadratic variation follows immediately from the definition. To show the second equality in (6.3) it is enough to notice that for any real numbers  $a, b$*

$$ab = \frac{1}{4}(a + b)^2 - \frac{1}{4}(a - b)^2$$

and to apply this identity to (6.2). Equation (6.4) follows easily from (6.2). ■

We shall find the quadratic variation of the  $d$ -dimensional Wiener process  $W$ . If  $i = j$  then using (6.2) and the result for one dimensional Wiener process  $W^i$  we find that

$$\langle W^i, W^i \rangle_t = t.$$

We shall show that for  $i \neq j$

$$\langle W^i, W^j \rangle_t = 0. \quad (6.6)$$

Note first that  $EV_n^{(2)}(W^i, W^j) = 0$  because  $W^i$  and  $W^j$  are independent. By definition

$$(V_n^{(2)}(W^i, W^j))^2 = \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} (W_{t_l^n}^i - W_{t_{l-1}^n}^i) (W_{t_m^n}^i - W_{t_{m-1}^n}^i) (W_{t_l^n}^j - W_{t_{l-1}^n}^j) (W_{t_m^n}^j - W_{t_{m-1}^n}^j).$$

Then independence of the increments and independence of Wiener processes  $W^i$  and  $W^j$  yield

$$\begin{aligned} E(V_n^{(2)}(W^i, W^j))^2 &= \sum_{l=1}^{k_n} E(W_{t_l^n}^i - W_{t_{l-1}^n}^i)^2 E(W_{t_l^n}^j - W_{t_{l-1}^n}^j)^2 \\ &= \sum_{l=1}^{k_n} (t_l^n - t_{l-1}^n)^2 \leq td(P^n) \end{aligned}$$

and the last expression tends to zero. Hence we proved that

$$\langle W \rangle_t = tI, \quad (6.7)$$

where  $I$  denotes the identity matrix. Consider an  $\mathbb{R}^m$ -valued process  $X_t = BW_t$ , where  $B$  is an arbitrary  $m \times d$  matrix and  $W$  is an  $\mathbb{R}^d$ -valued Wiener process. In future we shall develop other tools to investigate more general processes, but properties of this process can be obtained by methods already known.

First note that  $X$  is a linear transformation of a Gaussian process and hence is Gaussian itself. We can easily check that

$$EX_t = BEW_t = 0$$

and

$$\text{Cov}(X_s, X_t) = EX_s X_t^T = \min(s, t) BB^T. \quad (6.8)$$

The process  $X$  is also an  $(\mathcal{F}_t)$ -martingale. Indeed, for  $s \leq t$

$$E(X_t - X_s | \mathcal{F}_s) = BE(W_t - W_s | \mathcal{F}_s) = 0.$$

We will find quadratic variation of the process  $\langle X \rangle$  using (6.2):

$$\begin{aligned} V_n^{(2)}(X, X) &= \sum_{l=1}^{k_n} (X_{t_l} - X_{t_{l-1}}) (X_{t_l} - X_{t_{l-1}})^T \\ &= B \sum_{l=1}^{k_n} (W_{t_l} - W_{t_{l-1}}) (W_{t_l} - W_{t_{l-1}})^T B^T \\ &= BV_n^{(2)}(W, W) B^T. \end{aligned}$$

Therefore using (6.6) we obtain

$$\langle X \rangle_t = tBB^T. \quad (6.9)$$

In particular, if  $d = 1$  and  $X_t = bW_t$  then  $\langle X \rangle_t = b^2t$ .

Note that it follows from (6.7) and (6.8) that two coordinates of the process  $X_t$ ,  $X_t^i$  and  $X_t^j$  say, are independent if and only if  $\langle X^i, X^j \rangle_t = 0$ .

**Example 6.1** Let  $(W_t^1)$  and  $(W_t^2)$  be two independent Wiener processes and let

$$B_t^1 = aW_t^1 + bW_t^2$$

and

$$B_t^2 = cW_t^1 + dW_t^2.$$

We will find the joint quadratic variation  $\langle B^1, B^2 \rangle$ . Lemma 6.4 yields

$$\begin{aligned} \langle B^1, B^2 \rangle_t &= \langle aW^1 + bW^2, cW^1 + dW^2 \rangle_t \\ &= ac \langle W^1, W^2 \rangle_t + ad \langle W^1, W^2 \rangle_t + bc \langle W^1, W^2 \rangle_t + bd \langle W^1, W^2 \rangle_t \\ &= (ac + bd)t, \end{aligned}$$

where the last equality follows from (6.6).

## 6.1 EXERCISES

1. (Continuation of Example 6.1) Assume that  $a^2 + b^2 = 1$ . Show that in this case the process  $B^1$  is also a Wiener process. Find  $\langle W^1, B^1 \rangle_t$ .
2. Assume that  $W$  is a  $d$ -dimensional Wiener process and let  $U$  be a  $d \times d$  unitary matrix, that is  $U^T = U^{-1}$ . Show that the process  $X_t = UW_t$  is also a Wiener process.
3. Let  $W$  be a  $d$ -dimensional Wiener process.

- (a) For any  $a \in \mathbb{R}^d$  show that the process  $a^T W_t$  is a martingale

$$\langle a^T W \rangle_t = |a|^2 t,$$

where  $|a|$  is the length of the vector  $a$ .

- (b) Show that the process  $M_t = |W_t|^2 - td$  is a martingale.
- (c) Let  $X_t = BW_t$ , where  $B$  is an  $m \times d$  matrix. Show that the process

$$M_t = |X_t|^2 - \text{tr}(BB^T)t$$

is a martingale, where for any  $m \times m$  matrix  $C$

$$\text{tr}(C) = e_1^T C e_1 + \cdots + e_m^T C e_m$$

and  $e_1, \dots, e_m$  are basis vectors in  $\mathbb{R}^m$ , that is

$$e_i^T e_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Show also that the process

$$M_t^{i,j} = X_t^i X_t^j - \langle X^i, X^j \rangle_t$$

is a martingale for any choice of  $i, j \leq m$ . Deduce that the matrix-valued process  $X_t X_t^T - \langle X \rangle_t$  is a martingale.

4. Let  $W$  be a  $d$ -dimensional Wiener process and let  $X_t^x = |x + W_t|^2$ , where  $x$  is any starting point in  $\mathbb{R}^d$ . The process  $X_t^x$  is called a *Bessel process* starting from  $|x|^2$ . Using the definition of  $\chi^2$  distribution (or otherwise) write down the density of the random variable  $X_t^0$ . Using properties of the Wiener process show that the random variables  $X_t^x$  and  $X_t^{-x}$  have the same distribution.

# Chapter 7 STOCHASTIC INTEGRAL

Reference: Karatzas I. and Shreve S. E.: *Stochastic Calculus and Brownian Motion*

Let  $t$  be fixed and let  $P^n$  be any division of  $[0, t]$ . We define

$$M_n = \sum_{i=1}^{k_n} W_{t_{i-1}^n} (W_{t_i^n} - W_{t_{i-1}^n}).$$

Note first that by simple algebra

$$\begin{aligned} W_t^2 &= \left( \sum_{i=1}^{k_n} (W_{t_i^n} - W_{t_{i-1}^n}) \right)^2 \\ &= \sum_{i=1}^{k_n} (W_{t_i^n} - W_{t_{i-1}^n})^2 + 2 \sum_{i=1}^{k_n} W_{t_{i-1}^n} (W_{t_i^n} - W_{t_{i-1}^n}). \end{aligned}$$

Hence

$$M_n = \frac{1}{2} W_t^2 - \frac{1}{2} \sum_{i=1}^{k_n} (W_{t_i^n} - W_{t_{i-1}^n})^2$$

but the second term on the right hand side is known to converge to the quadratic variation of the Wiener process, and finally

$$\lim_{d(P^n) \rightarrow 0} E \left| M_n - \frac{1}{2} (W_t^2 - t) \right|^2 = 0.$$

Therefore we are able to determine the limit of the integral sums  $M^n$  which can justifiably be called the integral

$$\int_0^t W_s dW_s = \frac{1}{2} (W_t^2 - t).$$

In general the argument is more complicated but it repeats the same idea. For an arbitrary stochastic process  $X$  we define an integral sum

$$I_n = \sum_{i=1}^{k_n} X_{t_{i-1}^n} (W_{t_i^n} - W_{t_{i-1}^n})$$

determined by a division  $P^n$  of the interval  $[0, t]$ .

**Theorem 7.1** Assume that  $(X_t)$  is a process adapted to the filtration  $(\mathcal{F}_t)$  and such that

$$\int_0^t X_s^2 ds < \infty. \tag{7.1}$$

Then there exists an  $\mathcal{F}_t$ -measurable random variable

$$I = \int_0^t X_s dW_s$$

such that for every  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(|I_n - I| > \varepsilon) = 0.$$

If moreover,

$$\int_0^t EX_s^2 ds < \infty \tag{7.2}$$

then

$$\lim_{n \rightarrow \infty} E |I_n - I|^2 = 0.$$

If (7.1) holds for every  $t \leq T$  then the above theorem allows us to define an adapted stochastic process

$$M_t = \int_0^t X_s dW_s.$$

**Theorem 7.2** If  $(X_t)$  is an adapted process and

$$\int_0^T X_s^2 ds < \infty$$

then the stochastic integral enjoys the following properties.

1.  $M_0 = 0$ ;
2. for all  $0 \leq t_1 \leq t_2 \leq T$

$$\int_0^{t_1} X_s dW_s + \int_{t_1}^{t_2} X_s dW_s = \int_0^{t_2} X_s dW_s;$$

3. if  $(Y_t)$  is another adapted process such that

$$\int_0^T Y_s^2 ds < \infty$$

then for all  $t \leq T$

$$\int_0^t (aX_s + bY_s) dW_s = a \int_0^t X_s dW_s + b \int_0^t Y_s dW_s.$$

4. if moreover,

$$E \int_0^T X_s^2 ds < \infty$$

then the process

$$M_t = \int_0^t X_s dW_s$$

defined for  $t \leq T$  is a continuous martingale with respect to the filtration  $(\mathcal{F}_t)$  and

$$EM_t^2 = \int_0^t EX_s^2 ds.$$

It will be important to know what is the quadratic variation of the martingale defined by a stochastic integral. The answer is provided by the next theorem.

**Theorem 7.3** *If*

$$M_t = \int_0^t X_s dW_s$$

*then*

$$\langle M \rangle_t = \int_0^t X_s^2 ds.$$

**Corollary 7.4** *Let*

$$M_t = \int_0^t X_s dW_s^1 \quad \text{and} \quad N_t = \int_0^t Y_s dW_s^2,$$

*where*  $(Y_s), (X_s)$  *are two*  $(\mathcal{F}_t)$ -*adapted processes such that*

$$\int_0^t (X_s^2 + Y_s^2) ds < \infty,$$

*and*  $(W_t^1), (W_t^2)$  *are two*  $(\mathcal{F}_t)$ -*adapted Wiener processes. Then*

$$\langle M, N \rangle_t = \int_0^t X_s Y_s d\langle W^1, W^2 \rangle_s.$$

**Proof.** We prove the corollary for  $W^1 = W^2$ . The general case is left as an exercise. Invoking the result from chapter 5 we find that

$$\begin{aligned} \langle M, N \rangle_t &= \frac{1}{4} (\langle M + N, M + N \rangle_t - \langle M - N, M - N \rangle_t) \\ &= \frac{1}{4} \left( \int_0^t (X_s + Y_s)^2 ds - \int_0^t (X_s - Y_s)^2 ds \right) \\ &= \int_0^t X_s Y_s ds \end{aligned}$$

and the corollary follows. ■

Having defined a stochastic integral we can introduce a large class of processes called **semimartingales**. In these notes a semimartingale is a stochastic process  $(X_t)$  adapted to a given filtration  $(\mathcal{F}_t)$  such that

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s, \tag{7.3}$$

where  $X_0$  is a random variable measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_0$  and  $a, b$  are two adapted processes with the property

$$\int_0^T (|a_s| + b_s^2) ds < \infty.$$

We often write this process in a differential form

$$dX_t = a_t dt + b_t dW_t.$$

Equation (7.3) is called a decomposition of the semimartingale  $(X_t)$  into the finite variation part

$$A_t = \int_0^t a_s ds$$

and the martingale part

$$M_t = \int_0^t b_s dW_s.$$

For continuous semimartingales this decomposition is unique.

If

$$\int_0^T Y_s^2 b_s^2 ds < \infty \quad \text{and} \quad \int_0^T |Y_s a_s| ds < \infty$$

then we can define an integral  $\int Y_s dX_s$  of one semimartingale with respect to another:

$$\int_0^t Y_s dX_s = \int_0^t Y_s a_s ds + \int_0^t Y_s b_s dW_s, \quad t \leq T,$$

and the result is still a semimartingale. Because processes of finite variation have zero quadratic variation, the quadratic variation of a semimartingales  $(X_t)$  is the same as the quadratic variation of its martingale part:

$$\langle X \rangle_t = \langle M \rangle_t = \int_0^t b_s^2 ds.$$

It follows that, for any semimartingale  $(X_t)$ , if  $\langle X \rangle_t = 0$  for every  $t \geq 0$ , then its martingale part is zero and

$$X_t = X_0 + \int_0^t a_s ds.$$

If the process  $(X_t)$  adapted to the filtration  $(\mathcal{F}_t)$  is a semimartingale, then its decomposition (7.3) into a martingale and a process of bounded variation is called the **semimartingale representation** of  $X_t$ .

We consider now  $d$  semimartingales adapted to the same filtration  $(\mathcal{F}_t)$

$$X_t^i = X_0^i + A_t^i + M_t^i = X_0^i + \int_0^t a_s^i ds + \int_0^t b_s^i dW_s^i, \quad (7.4)$$

where  $i = 1, \dots, d$  and  $(W_t^i)$  are  $(\mathcal{F}_t)$ -Wiener processes.

**Theorem 7.5 (Ito's formula)** *Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  be a function with two continuous derivatives and let the semimartingales  $(X_t^1), \dots, (X_t^d)$  be given by (7.4). Then*

$$\begin{aligned} F(X_t^1, \dots, X_t^d) &= F(X_0^1, \dots, X_0^d) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^d) dA_s^i \\ &\quad + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^d) dM_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \dots, X_s^d) d\langle X^i, X^j \rangle_s. \end{aligned}$$

In a more explicit form the Ito's formula can be written as

$$\begin{aligned} F(X_t^1, \dots, X_t^d) &= F(X_0^1, \dots, X_0^d) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^d) a_s^i ds \\ &\quad + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^d) b_s^i dW_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \dots, X_s^d) b_s^i b_s^j d\langle W^i, W^j \rangle_s. \end{aligned}$$

**Corollary 7.6** Let  $X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$  and let  $F : [0, \infty] \times \mathbb{R} \rightarrow \mathbb{R}$  have two continuous derivatives. Then

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) a_s ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) a_s ds \\ &\quad + \int_0^t \frac{\partial F}{\partial x}(s, X_s) b_s dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) b_s^2 ds. \end{aligned}$$

**Proof.** It is enough to put in Theorem 7.5  $d = 2$ ,  $X_t^1 = t$  and  $X_t^2 = X_t$ . ■

It is often convenient to write the Ito's formula in its infinitesimal form

$$dF(t, X_t) = \left( \frac{\partial F}{\partial t}(t, X_t) + a_t \frac{\partial F}{\partial x}(t, X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 F}{\partial x^2}(t, X_t) \right) dt + b_t \frac{\partial F}{\partial x}(t, X_t) dW_t.$$

**Example 7.1** Let  $X_t = W_t^n$ . Then

$$X_t = \int_0^t n W_s^{n-1} dW_s + \int_0^t \frac{1}{2} n(n-1) W_s^{n-2} ds = n \int_0^t W_s^{n-1} dW_s + \frac{n(n-1)}{2} \int_0^t W_s^{n-2} ds$$

and therefore

$$\int_0^t W_s^{n-1} dW_s = \frac{1}{n} W_t^n - \frac{n-1}{2} \int_0^t W_s^{n-2} ds. \quad (7.5)$$

For  $n = 2$  we recover in a simple way the fact already known that

$$\int_0^t W_s dW_s = \frac{1}{2} (W_t^2 - t).$$

Let

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s^1$$

and

$$Y_t = Y_0 + \int_0^t u_s ds + \int_0^t v_s dW_s^2$$

be two semimartingales with possibly dependent Wiener processes ( $W_t^1$ ) and ( $W_t^2$ ). We will find the semimartingale representation of the process  $Z_t = X_t Y_t$ . The Ito formula applied to the function  $F(x, y) = xy$  yields immediately

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t (X_s u_s + Y_s a_s) ds \\ &\quad + \int_0^t X_s v_s dW_s^2 + \int_0^t Y_s b_s dW_s^1 + \int_0^t b_s v_s d\langle W^1, W^2 \rangle_s. \end{aligned}$$

We can write this formula in a compact form

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y \rangle_t$$

which can be called a stochastic integration by parts formula.

For

$$M_t = \int_0^t X_s dW_s,$$

consider the process

$$N_t = \exp \left( M_t - \frac{1}{2} \langle M \rangle_t \right).$$

We will apply the Ito formula to the process  $F(X_t^1, X_t^2)$ , where

$$X_t^1 = \langle M \rangle_t = \int_0^t X_s^2 ds, \quad X_t^2 = M_t = \int_0^t X_s dW_s$$

and  $F(x_1, x_2) = e^{x_2 - \frac{1}{2}x_1}$ . Then

$$\begin{aligned} \frac{\partial F}{\partial x_1}(x_1, x_2) &= -\frac{1}{2}F(x_1, x_2), \\ \frac{\partial F}{\partial x_2}(x_1, x_2) &= \frac{\partial^2 F}{\partial x_2^2}(x_1, x_2) = F(x_1, x_2) \end{aligned}$$

and therefore the Ito formula yields

$$N_t = 1 + \int_0^t \left( -\frac{1}{2} \right) N_s X_s^2 ds + \int_0^t N_s X_s dW_s + \frac{1}{2} \int_0^t N_s X_s^2 ds = 1 + \int_0^t N_s X_s dW_s.$$

As a by-product we find that

$$\langle N \rangle_t = \int_0^t N_s^2 X_s^2 ds.$$

The process  $N$  will be important in many problems. Below are the first applications.

**Example 7.2** Let  $S$  be an exponential Wiener process

$$S_t = S_0 \exp(mt + \sigma W_t).$$

Clearly

$$S_t = S_0 e^{mt + \frac{1}{2}\sigma^2 t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t} = S_0 e^{mt + \frac{1}{2}\sigma^2 t} N_t.$$

In this case, using the example from Lecture 5 we find that

$$\sigma^2 \int_0^T E N_s^2 ds < \infty$$

and therefore the process  $N$  is a martingale. Hence we obtain the semimartingale representation of the exponential Wiener process

$$S_t = S_0 + \int_0^t \left( m + \frac{1}{2}\sigma^2 \right) S_s ds + \int_0^t \sigma S_s dW_s$$

and if  $m = -\frac{1}{2}\sigma^2$  then, as we already know,  $(S_t)$  is a martingale with the representation

$$S_t = S_0 + \int_0^t \sigma S_s dW_s.$$

In financial applications the coefficient  $m$  is usually written in the form

$$m = r - \frac{1}{2}\sigma^2$$

and then

$$S_t = S_0 + \int_0^t r S_s ds + \int_0^t \sigma S_s dW_s.$$

Let us consider a simple but important case of a stochastic integral with the deterministic integrand  $f$  such that

$$\int_0^T f^2(s) ds < \infty.$$

It follows from Theorem 7.2 that the process

$$M_t = \int_0^t f(s) dW_s. \quad t \leq T,$$

is a continuous martingale but in this case we can say more.

**Proposition 7.7** *Let*

$$M_t = \int_0^t X_s dW_s.$$

The process  $(M_t)$  is Gaussian if and only if its quadratic variation process

$$\langle M \rangle_t = \int_0^t X_s^2 ds$$

is deterministic or, equivalently  $(X_s)$  is a deterministic function.

**Theorem 7.8 (Levy Theorem)** *Let  $(M_t)$  be an  $\mathbb{R}^d$ -valued continuous  $(\mathcal{F}_t)$ -martingale starting from zero with the quadratic variation process  $\langle M \rangle = (\langle M^{ij} \rangle)$ . Then  $(M_t)$  is a Wiener process if and only if*

$$\langle M^{ij} \rangle_t = \begin{cases} t & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (7.6)$$

**Proof.** If  $(W_t)$  is a Wiener process then its quadratic variation is given by (7.6). Assume now that the process  $(M_t)$  satisfies the assumptions of the theorem. We need to show that  $(M_t)$  is a Wiener process. We shall apply the theorem from chapter 2. Let, for every  $a \in \mathbb{R}^d$ ,

$$F(a, x) = e^{ia^T x}$$

be a function defined on  $\mathbb{R}^d$ . Then

$$\frac{\partial F}{\partial x_j}(a, x) = ia_j F(a, x), \quad \frac{\partial^2 F}{\partial x_j \partial x_k}(a, x) = -a_j a_k F(a, x).$$

By the Ito formula

$$F(a, M_t) = e^{ia^T M_s} + i \sum_{j=1}^d a_j \int_s^t e^{ia^T M_u} dM_u - \frac{1}{2} \sum_{i,j=1}^d a_j^2 \int_s^t e^{ia^T M_u} du.$$

Hence

$$\begin{aligned} & e^{ia^T (M_t - M_s)} \\ &= 1 + i \sum_{j=1}^d a_j \int_s^t e^{ia^T (M_u - M_s)} dM_u - \frac{1}{2} \sum_{i,j=1}^d a_j^2 \int_s^t e^{ia^T (M_u - M_s)} du. \end{aligned} \quad (7.7)$$

By the properties of stochastic integrals

$$E \left( \int_s^t e^{ia^T M_u} dM_u \mid \mathcal{F}_s \right) = 0.$$

Let  $A \in \mathcal{F}_s$ . Then

$$E \left( e^{ia^T (M_t - M_s)} I_A \right) = EI_A - \frac{1}{2} |a|^2 \int_s^t E \left( e^{ia^T (M_u - M_s)} I_A \right) du.$$

If we denote

$$f(t) = E \left( e^{ia^T (M_t - M_s)} I_A \right)$$

then we obtain an equation

$$f(t) = P(A) - \frac{|a|^2}{2} \int_s^t f(u) du$$

or

$$f'(t) = -\frac{|a|^2}{2} f(t)$$

with the initial condition  $f(s) = P(A)$ . It is easy to show that the unique solution to this differential equation is given by the formula

$$f(t) = P(A) \exp \left( -\frac{|a|^2}{2} (t - s) \right).$$

Finally, taking into account the definition of conditional expectation we proved that

$$E \left( e^{ia^T (M_t - M_s)} I_A \right) = E \left( I_A E \left( e^{ia^T (M_t - M_s)} \mid \mathcal{F}_s \right) \right) = E \left( I_A \exp \left( -\frac{|a|^2}{2} (t - s) \right) \right)$$

and therefore

$$E \left( e^{ia^T (M_t - M_s)} \mid \mathcal{F}_s \right) = e^{-|a|^2(t-s)/2}$$

and therefore the random variable  $(M_t - M_s)$  is independent of  $\mathcal{F}_s$  and has the normal  $N(0, (t - s)I)$  distribution and the proof is finished. ■

We will apply this theorem to quickly show the following:

**Corollary 7.9** *Let  $(W_t)$  be an  $\mathbb{R}^d$ -valued Wiener process and let the matrix  $B$  be unitary:  $B^T = B^{-1}$ . Then the process  $X_t = BW_t$  is also an  $(\mathcal{F}_t)$ -Wiener process.*

**Proof.** We already know that  $X$  is a continuous martingale with quadratic variation

$$\langle X \rangle_t = tBB^T.$$

Now, by assumption  $\langle X \rangle_t = tI$  and the Levy Theorem concludes the proof. ■

## 7.1 EXERCISES

1. Find the semimartingale representation of the process

$$X_t = e^{at}x + e^{at} \int_0^t e^{-as}b dW_s,$$

where  $a, b$  are arbitrary constants. Show that

$$X_t = x + a \int_0^t X_s ds + bW_t.$$

2. Use the Ito's Formula to write a semimartingale representation of the process

$$Y_t = \cos(W_t).$$

Next, apply the properties of stochastic integrals to show that the function  $m(t) = EY_t$  satisfies the equation

$$m'(t) = 1 - \frac{1}{2} \int_0^t m(s) ds.$$

Argue that  $m'(t) = -\frac{1}{2}m(t)$  and  $m(0) = 1$ . Show that  $m(t) = e^{-t/2}$ .

3. Let

$$X_t = \int_0^t b_s dW_s,$$

where

$$b_t = \begin{cases} 1 & \text{if } W_t \geq 0, \\ -1 & \text{if } W_t < 0. \end{cases}$$

Show that  $X_t$  is a Wiener process.

4. Let  $(W_t)$  be an  $(\mathcal{F}_t)$ -Wiener process and let  $F : [0, \infty] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the assumptions of the Ito's Lemma. Find conditions on the function  $F$  under which the process  $Y_t = F(t, W_t)$  is a martingale.
5. Prove the general version of Corollary 7.4.