Investments and valuation when early exercise is allowed

An integrated approach

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Related papers


- A valuation algorithm for early exercise claims in incomplete markets (with E. Sokolova, preprint 2005)

- The spot and forward utilities and the associated pricing systems: Case study of the binomial model (with M. Musiela, 2003, 2004)
Contents

• Optimal portfolio choice with discretionary stopping
• Integrated models in incomplete markets
• Optimal behavior and pricing under exponential preferences
• A valuation algorithm for early exercise claims
• Modelling issues
Optimal portfolio choice with discretionary stopping
Stochastic setting

Market $\leftrightarrow$ Preferences

Objective

Market:
\[ dS_s = \mu_s S_s \, ds + \sigma_s S_s \, dW_s \quad ; \quad (\Omega, \mathcal{F}, \mathbb{P}) \]
\[ dX_s = \mu_s \pi_s \, ds + \sigma_s \pi_s \, dW_s \]

Preferences: \( U \), concave and increasing functional

Objective: \( V_t = \sup_{\pi, \tau} \mathbb{E}_{\mathbb{P}}[U(X_{\tau}; C, \omega) / \mathcal{F}_t] \), \quad t \in [0, T]
Complete markets

Primal utility problem with discretionary stopping

Dual problem of pricing an early exercise claim

- General market setting
- Abstract results
- Difficult to retrieve optimal portfolios and optimal stopping time

Karatzas-Wang, El Karoui-Jeanblanc, Hugonnier-Morellec
Incomplete markets

Very limited results

Difficulties

- Duality “fails”
- No closed form solutions
- Variational problems degenerate, fully nonlinear, high-dimensional

Davis-Z., Scheinkman-Z., Rogers-Scheinkman, Hann, Haugh et al.
Diffusion models

Hamilton-Jacobi-Bellman + obstacle

Value function \( V(x, S, y, t) = \sup_{\pi, \tau} E_P[U(X_\tau, S_\tau, Y_\tau) / \mathcal{F}_t] \), \( t \in [0, T] \)

Weak sense

\[
\min \left\{ -V_t - \max_\pi (F(V_x, V_xS, V_{xy}, V_{xx}; \pi) - \mathcal{L}(y,S)V, V - U) \right\} = 0 \quad \text{(HJB–FBP)}
\]

portfolio choice

stochastic opportunity set

discretionary stopping

\[
F(\nabla V, \Delta V; \pi) = \frac{1}{2} \sigma^2 \pi^2 V_{xx} + \pi (\mu V_x + \sigma^2 S V_{xS} + \rho \sigma a V_{xy})
\]
Results

Under mild conditions, $V$ is the **unique** (constrained) **viscosity** solution of (HJB-FBP)

Numerical approximations

Regularities

Schemes

- Montone
- Consistent
- Robust

Open

- Reduction to quasilinear
- or linear problems for
- special cases
Integrated models
Optimal behavior when a claim/contract is incorporated

• **Objective**

\[
V^{w,b}(X_t, S_t, Y_t, t) = \sup_{\pi, \tau} E_{\mathbb{P}} [U(X_{\tau} \pm G(S_{\tau}, Y_{\tau})) / \mathcal{F}_t]
\]

\[G: \text{ claim}\]

• **Asymmetries**  w.r.t. writer’s and buyer’s optimality criteria, features of the contract etc.
Complete markets

Duality and variational methods

\[ V(X_s, S_s, Y_s, s) = V^0(X_s \pm h_s, S_s, Y_s, s) \]

\[ s \in [t, T] \]

\[ h_s = \sup_\tau E_Q\left( \frac{G(S_\tau, Y_\tau)}{\mathcal{F}_s} \right) \]

\[ \pi^*_s = \pi^0_s + \sigma S_s \frac{\partial h_s}{\partial S} \]

Even if the market is complete several issues/problems arise with regards to trading horizon, numeraires etc.
Incomplete markets

Partial equilibrium approach

No claim: \( V^0(X_t, S_t, Y_t, t) = \sup_{\pi, \tau} E_{\mathbb{P}}(U(X_\tau)/\mathcal{F}_t) ; \quad t \in [0, T] \)

Claim: \( V^G(X_t, S_t, Y_t, t) = \sup_{\pi, \tau} E_{\mathbb{P}}(U(X_\tau - G(S_\tau, Y_\tau))/\mathcal{F}_t) \)

Indifference price: \( h_t(G) \)

\[ V^0(X_t, S_t, Y_t, t) = V^G(X_t + h_t(G), S_t, Y_t, t) \]

Davis-Z., Musiela-Z., Kallsen-Kuhn, Oberman-Z., Henderson, Ilhan, Lau-Kwok,...
An example

- $dS_t = \mu(S_t) S_t \, dt + \sigma(S_t) S_t \, dW_t$ ; $\lambda_t = \frac{\mu(S_t)}{\sigma(S_t)}$, $t \in [0, T]$

- $dX_t = \mu(S_t) \pi_t \, dt + \sigma(S_t) \pi_t \, dW_t$

- $U(x) = -e^{-\gamma x}$, $\gamma > 0$ \hspace{1cm} $G = G(S_{\tau})$

- $V^0(X_t, S_t, t) = \sup_{\pi, \tau} E_{\mathbb{P}}(-e^{-\gamma X_{\tau}} / \mathcal{F}_t)$

  \[= - \exp \left( -\gamma \left( X_t + \sup_{\tau} E_{\mathbb{Q}} \left( \int_t^\tau \frac{1}{2\gamma} \lambda^2(S_s) \, ds / \mathcal{F}_t \right) \right) \right)\]

  \[= - \exp \left( -\gamma \left( X_t + E_{\mathbb{Q}} \left( \int_t^T \frac{1}{2\gamma} \lambda^2(S_s) \, ds / \mathcal{F}_t \right) \right) \right)\]

- It is never optimal to stop!
\[ V^G(X_t, S_t, t) = \sup_{\pi, \tau} E_{\mathbb{P}}(-e^{-\gamma(X_\tau - G(S_\tau))}/\mathcal{F}_t) \]

\[ = -e^{-\gamma(X_t + \sup_{\tau} E_{Q} \left( \int_t^\tau \frac{1}{2\gamma} \lambda^2(S_s) \, ds + G(S_\tau)/\mathcal{F}_t \right) } \]

\[ \text{Indifference price} \]

\[ h_t = \sup_{\tau} E_{Q} \left( \int_\tau^T \frac{1}{2\gamma} \lambda^2(S_s) \, ds + G(S_\tau)/\mathcal{F}_t \right) \]

An obviously wrong result!
WWW  (what went wrong)

Preferences were wrongly specified!

\[ V^G(X_t, S_t, Y_t, t) = \begin{cases} 
\neq \sup_{\pi, \tau} \left( U(X_\tau - G(S_\tau))/\mathcal{F}_t \right) \\
= \sup_{\pi, \tau} E_{\mathbb{P}} \left( V^0(X_\tau - G(S_\tau), \tau)/\mathcal{F}_t \right) 
\end{cases} \]

Davis-Z., Oberman-Z., Musiela-Z., Henderson, Evans et al.
Optimal behavior under exponential risk preferences in incomplete markets
**Diffusion models**

- **Simple incomplete model**  (no internal incompleteness)

  Traded asset:  
  \[ dS_t = \mu(S_t)S_t \, dt + \sigma(S_t)S_t \, dW_t^1 \]

  Non-traded factor:  
  \[ dY_t = b(Y_t) \, dt + a(Y_t) \, dW_t \]

  \[ \rho = \text{cor}(W^1, W) \in (-1, 1) \]

  Claim:  
  \[ G(S_{\tau}, Y_{\tau}) \]

  \[ \downarrow \]

  Indifference price FBP

  \[ \min \left\{ -h_t - \mathcal{L}^{(S,y)} h + (b(y) - \rho \frac{\mu}{\sigma} a(y)) h_y - \frac{1}{2} \gamma (1 - \rho^2) a^2(y) h^2_y, \quad h - G(S, y) \right\} = 0 \]

  Musiela-Z., Oberman-Z.
• **More involved** incomplete model  (internal incompleteness)

Traded asset: \[ dS_t = \mu(Y_t)S_t \, dt + \sigma(Y_t)S_t \, dW_t^1 \]

Non-traded factor: \[ dY_t = b(Y_t) \, dt + a(Y_t) \, dW_t \]

\[ \downarrow \]

Indifference price FBP

\[
\min \left\{ -h_t - \mathcal{L}^{(S,y)} h - \left( b(y) - \rho \frac{\mu}{\sigma} a(y) + a^2(y) \frac{u_y(y, t; T)}{u(y, t; T)} \right) h_y \\
-\frac{1}{2} \gamma(1 - \rho^2)a^2(y)h_y^2 , \ h - G(S, y) \right\} = 0
\]

The function \( u \) solves a **linear** pde; the drift displacement is associated with the minimal relative entropy measure (Benth-Karlsen, Stoikov-Z.)
Indifference price representation
European case

\[ \nu_t^e = \mathcal{E}_{Q}^{(t,T)}(G(S_T, Y_T)) \]

where \( \mathcal{E} \) is the relevant non-linear “expectation” functional and \( Q \) the mem

Example:

\[ G(S, y) = G(y) \]

\[ \mathcal{E}_{Q}^{(t,T)}(G) = \frac{1}{\gamma(1 - \rho^2)} \ln E_Q \left( e^{\gamma(1 - \rho^2)G(Y_T)/Y_t = y} \right) \]

Non-linear martingale property

\[ \nu_t^e = \mathcal{E}_{Q}^{(t,s)}(\nu_s^e) \]

with

\[ \nu_t^e = H(S_t, Y_t, t) \]

\( H \) solves a quasilinear pde
Early exercise claims

The American indifference price can be represented in terms of its European indifference counterpart in a manner consistent with the complete market case

\[ \nu^a_t = \sup_{\tau} \mathcal{E}_{Q}^{(t,\tau)}(\nu^e_\tau) \]

Musiela-Z., Sokolova-Z.
**Issues**

- **Early exercise premium:**
  \[ e_t = \nu_t^a - \nu_t^e \]
  The relevant pde has the same structure as IP–FBP but with different obstacle (Arnarson-Z.)

- **Price decomposition:**
  \[ G_T = \nu_t^a + L_{t,\tau} + R_{t,\tau} \quad t \in (s, T] \]
  \[ \downarrow \quad \downarrow \]
  replicable part  residual risk

  \[ \nu_t^a(R_{t,\tau}) = 0 \]
A valuation algorithm for early exercise claims
Binomial approximations (Sokolova-Z., 2005)

The multiperiod model

• **Traded asset:**  \( S_t, \ t = 0, 1, ..., T \)  \( (S_t > 0, \ \forall t) \)

\[
\xi_{t+1} = \frac{S_{t+1}}{S_t}, \ \xi_{t+1} = \xi^d_{t+1}, \ \xi^u_{t+1} \quad \text{with} \quad 0 < \xi^d_{t+1} < 1 < \xi^u_{t+1}
\]

Second traded asset is riskless yielding zero interest rate

• **Nontraded asset:**  \( Y_t, \ t = 0, 1, ..., T \)

\[
\eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \ \eta_{t+1} = \eta^d_{t+1}, \ \eta^u_{t+1} \quad \text{with} \quad \eta^d_t < \eta^u_t
\]

\( \{S_t, Y_t : t = 0, 1, ..., T\} : \) a two-dimensional stochastic process

• **Probability space:**  \( (\Omega, (\mathcal{F}_t), \mathbb{P}) \)

Filtrations  \( \mathcal{F}^S_t \) and  \( \mathcal{F}^Y_t \) : generated by the random variables  \( S_s (\xi_s) \)
and  \( Y_s (\eta_s), \) for  \( s = 0, 1, \ldots, t. \)
• Investment problem without the liability:

\[ V^0(X_t, t) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} E(U(X_T)/\mathcal{F}_t), \quad 0 < t < T \]

• Investment problem with the liability:

\[ V^G(X_t, Y_t, t) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} \sup_{t \leq \tau \leq T} E[V^0(X_\tau + G_\tau, \tau)/\mathcal{F}_t] \]

• American indifference price: \( \nu^a_t(G) \)

\[ V^0(X_t, t) = V^G(X_t - \nu^a_t(G), Y_t, t) \]
• **Pricing measure** $ℚ$ is a martingale measure that preserves the conditional distribution

$$ℚ\left(\eta_{t+1} / ℱ_t ∨ ℱ_{t+1}^S\right) = ℙ\left(\eta_{t+1} / ℱ_t ∨ ℱ_{t+1}^S\right)$$

• The **one-period European** pricing operator:

$$𝔼^{(s-1,s)}_{ℚ}(Z_s) = -𝔼_{ℚ}\left[\frac{1}{\gamma} \ln ℋ_{ℚ}\left[ e^{-\gamma Z_s / ℱ_{s-1} ∨ ℱ_s^S} \right] / ℱ_{s-1}\right]$$

• The **one-period American** pricing operator:

$$𝒜^{(s-1,s)}_{ℚ}(Z_s) = \max\{G_{s-1}, ℋ^{(s-1,s)}_{ℚ}(Z_s)\}$$

• The **multi-period American** pricing operator:

$$\begin{align*}
\mathcal{A}^{(s,s)}_{ℚ}(Z_s) &= Z_s, \\
\mathcal{A}^{(t,s)}_{ℚ}(Z_s) &= \mathcal{A}^{(t,s-1)}_{ℚ}(\mathcal{A}^{(s-1,s)}_{ℚ}(Z_s))
\end{align*}$$
The indifference price process $\nu_t^a(G)$ satisfies:

$$(i) \quad \begin{cases} 
\nu_t^a(G) = \max \left\{ G_t, \mathcal{E}_{Q}^{(t,t+1)}(\nu_{t+1}^a(G)) \right\}, & t < T \\
\nu_T^a(G) = G_T,
\end{cases}$$

where $\mathcal{E}_{Q}^{(t,t+1)}$ is the one-period European pricing operator.

$$(ii) \quad \nu_t^a(G) = A_{Q}^{(t,T)}(G_T),$$

where $A_{Q}^{(t,T)}$ is the multi-period American pricing operator.

$$(iii) \quad \text{The \ semigroup\ property\ holds:}$$

$$\nu_t^a(G) = A_{Q}^{(t,s)}(A_{Q}^{(s,T)}(G_T)) = A_{Q}^{(t,s)}(\nu_s^a(G)) = \nu_t^a(A_{Q}^{(s,T)}(G_T)),$$

for $0 \leq t \leq s \leq T$. 
Modelling issues
Risk preferences and market environment

- Proper specification of “current” maximal utility
- Risk aversion specification
- Consistency across units

\[ U(x, \omega) = -e^{-\gamma T X_T} \]

Question: If we allow for “stochastic views in risk aversion”, are there internal modelling constraints?