Investment performance measurement, risk tolerance and optimal portfolio choice

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References

- Indifference valuation in binomial models (with M. Musiela)
- Investments and forward utilities (with M. Musiela)
- Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model (with M. Musiela)
- Optimal asset allocation under forward exponential criteria (with M. Musiela)
- Horizon-independent risk measures (with G. Zitkovic)
Performance measurement
Performance process

\(U_t(x)\) is an \(\mathcal{F}_t\)-adapted process

- As a function of \(x\), \(U\) is increasing and concave

- For each self-financing strategy, represented by \(\pi\), the associated (discounted) wealth \(X_t\) satisfies

  \[E_{\mathbb{P}}(U_t(X_t^\pi) \mid \mathcal{F}_s) \leq U_s(X_s^\pi) \quad 0 \leq s \leq t\]

- There exists a self-financing strategy, represented by \(\pi^*\), for which the associated (discounted) wealth \(X_t^{\pi^*}\) satisfies

  \[E_{\mathbb{P}}(U_t(X_t^{\pi^*}) \mid \mathcal{F}_s) = U_s(X_s^{\pi^*}) \quad 0 \leq s \leq t\]
Traditional framework

A deterministic utility datum \( u_T(x) \) is assigned at the end of a fixed investment horizon

\[
U_T(x) = u_T(x)
\]

Backwards in time generation of optimal utility

\[
V_t(x) = \sup_{\pi} E_{\pi}(u_T(X_T^{\pi})|\mathcal{F}_t; X_t^{\pi} = x)
\]

\[
V_t(x) = \sup_{\pi} E_{\pi}(V_s(X_s^{\pi})|\mathcal{F}_t; X_t^{\pi} = x) \quad \text{(DPP)}
\]

\[
V_t(x) = E_{\pi}(V_s(X_s^{\pi*})|\mathcal{F}_t; X_t^{\pi*} = x)
\]

\[
\downarrow
\]

\[
U_t(x) \equiv V_t(x) \quad 0 \leq t < T
\]

The performance process coincides with the traditional value function
Alternative framework

A deterministic datum \( u_0(x) \) is assigned at the beginning of the trading horizon, \( t = 0 \)

\[
U_0(x) = u_0(x)
\]

Forward in time generation of optimal performance

\[
U_s(X_{s}^{\pi^*}) = E_{\mathbb{P}}(U_t(X_{t}^{\pi^*})|\mathcal{F}_s) \quad 0 \leq s \leq t
\]

- Performance can be defined for all trading horizons
- Performance and allocations take a very intuitive form
- Difficulties due to the “inverse in time” nature of the problem

**Performance criterion is not exogenously given but is implied/calibrated w.r.t. investment opportunities**
Motivational examples
An incomplete multiperiod binomial example

Exponential datum

- **Traded security:** $S_t$, $t = 0, 1, ...$

  \[ \xi_{t+1} = \frac{S_{t+1}}{S_t}, \quad \xi_{t+1} = \xi_d, \xi_u \quad \text{with} \quad 0 < \xi_d < 1 < \xi_u \]

  Second traded asset is riskless yielding zero interest rate

- **Stochastic factor:** $Y_t$, $t = 0, 1, ...$

  \[ \eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \quad \eta_{t+1} = \eta_d, \eta_u \quad \text{with} \quad \eta_d < \eta_u \]

- **Probability space** $(\Omega, (\mathcal{F}_t), \mathbb{P})$

  \( \{S_t, Y_t : t = 0, 1, ...\} \) : a two-dimensional stochastic process
• **State wealth process**: $X_t, t = s + 1, s + 2, \ldots, \ldots$

$\alpha_i$: the number of shares of the traded security held in this portfolio over the time period $[i - 1, i]$

$$X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \triangle S_i$$

• **Forward exponential performance**

$$\begin{cases} 
U_s(X_s^{\alpha^*}) = E_{\mathbb{P}}(U_t(X_t^{\alpha^*})|\mathcal{F}_s) \\
U_0(x) = -e^{-\gamma x}, \quad \gamma > 0
\end{cases}$$
• A forward performance process

\[ U_t(x) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = 0 \\
-e^{-\gamma x} + \sum_{i=1}^{t} h_i & \text{if } t \geq 1 
\end{cases} \]

• Auxiliary quantities

\[ h_i = q_i \log \frac{q_i}{\mathbb{P}(A_i | \mathcal{F}_{i-1})} + (1 - q_i) \log \frac{1 - q_i}{1 - \mathbb{P}(A_i | \mathcal{F}_{i-1})} \]

with

\[ A_i = \{ \xi_i = \xi_i^u \} \quad \text{and} \quad q_i = \mathbb{Q}(A_i | \mathcal{F}_{i-1}) \]

for \( i = 0, 1, \ldots \) and \( \mathbb{Q} \) being the minimal martingale measure
Important insights

The forward performance process

\[ U_t(x) = -e^{-\gamma x + \sum_{i=1}^{t} h_i} \]

is of the form

\[ U_t(x) = u(x, A_t) \]

where \( u(x, t) \) is the deterministic function

\[ u(x, t) = -e^{-\gamma x + \frac{1}{2} t} \]

and \( A_t \) corresponds to a time change depending on the “market input”

\[ A_t = 2 \sum_{i=1}^{t} h_i \]
A continuous-time example

Arbitrary datum

- **Investment opportunities**

  Riskless bond: \( r = 0 \)

  Risky security: \( dS_t = \sigma_t S_t (\lambda_t dt + dW_t) \)

- **Datum at** \( t = 0 \): \( u_0(x) \) concave, increasing

- **Wealth process**

  \[
  \begin{align*}
  dX_t &= \sigma_t \pi_t (\lambda_t dt + dW_t) \\
  X_0 &= x
  \end{align*}
  \]

- **Market input**: \( \lambda_t, A_t \)

  \[
  \begin{align*}
  dA_t &= \lambda_t^2 dt \\
  A_0 &= 0
  \end{align*}
  \]
• Building the martingale $U_t(X_t^{\pi^*})$

Assume that we can construct $U_t(x)$ via

$$U_t(X_t^{\pi^*}) = u(X_t^{\pi^*}, A_t)$$

where $u(x, t)$ is the differential utility input and $A_t$ the stochastic market input

$$dU_t(X_t^{\pi}) = u_x(X_t, A_t)\sigma_t \pi_t dW_t$$

$$+ (u_t(X_t^{\pi}, A_t)\lambda_t^2 + u_x(X_t^{\pi}, A_t)\sigma_t \pi_t \lambda_t + \frac{1}{2}u_{xx}(X_t^{\pi}, A_t)\sigma_t^2 \pi_t^2)dt$$

$$\parallel \alpha_t = \lambda_t^{-1} \sigma_t \pi_t$$

$$+ \lambda_t^2 (u_t(X_t^{\pi}, A_t) + u_x(X_t^{\pi}, A_t)\alpha_t + \frac{1}{2}u_{xx}(X_t^{\pi}, A_t)\alpha_t^2)dt$$
• Differential utility input condition

\begin{equation}
\begin{cases}
    u_t u_{xx} = \frac{1}{2} u_x^2 \\
    u(x, 0) = u_0(x)
\end{cases}
\end{equation}

• The optimal allocations in stock, $\pi_t^*$, and in bond, $\pi_t^{0,*}$,

\begin{equation}
\begin{cases}
    \pi_t^* = -\sigma_t^{-1} \lambda_t \frac{u_x(X_t^{\pi^*}, A_t)}{u_{xx}(X_t^{\pi^*}, A_t)} = \sigma_t^{-1} \lambda_t R_t \\
    \pi_t^{0,*} = X_t^{\pi^*} - \sigma_t^{-1} \lambda_t R_t
\end{cases}
\end{equation}

\[ R_t = r(X_t^{\pi^*}, A_t) ; \quad r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)} \]

The local risk tolerance $r(x, t)$ and the subordinated risk tolerance process $R_t$ emerge as important quantities
Forward performance measurement

time $t_1$, information $\mathcal{F}_{t_1}$

- asset returns
- constraints
- market view
- away from equilibrium
- benchmark numeraire
- calendar time subordination

$MI(t_1) \rightarrow + \downarrow u(x, t_1)$

$U_{t_1}(x; MI) \in \mathcal{F}_{t_1}$, $\pi_{t_1}(x; MI) \in \mathcal{F}_{t_1}$
Forward performance measurement

time $t_2$, information $\mathcal{F}_{t_2}$

asset returns
constraints
market view
away from equilibrium
benchmark numeraire
calendar time subordination

$MI(t_2) \quad \rightarrow \quad + \quad \leftarrow \quad u(x, t_2)$

$U_{t_2}(x; MI) \in \mathcal{F}_{t_2}$
$\pi_{t_2}(x; MI) \in \mathcal{F}_{t_2}$
Forward performance measurement

time $t_3$, information $\mathcal{F}_{t_3}$

asset returns
constraints
market view
away from equilibrium
benchmark numeraire
calendar time subordination

$MI(t_3) \quad \rightarrow \quad + \quad \downarrow \quad \leftarrow \quad u(x, t_3)$

$U_{t_3}(x; MI) \in \mathcal{F}_{t_3} \quad \pi_{t_3}(x; MI) \in \mathcal{F}_{t_3}$
Forward performance measurement

time $t$, information $\mathcal{F}_t$

$MI(t)$ $\rightarrow$ $+$ $\leftarrow u(x,t)$

$U_t(X^*_t) \in \mathcal{F}_t$  $\pi_t^*(X^*_t) \in \mathcal{F}_t$
Construction of a class of forward performance processes
Creating the martingale that yields the optimal performance

Minimal model assumptions

Stochastic optimization problem “inverse” in time

Key idea

Stochastic input

Market

Differential input

Individual

Maximal performance — Optimal allocation
Differential input – utility surfaces
Performance surface

A model independent differential constraint on impatience, risk aversion and monotonicity

- Initial datum

\[ u_0(x) = u(x, 0) \]

- Fully non-linear pde

\[
\begin{cases}
  u_t u_{xx} = \frac{1}{2} u_x^2 \\
  u(x, 0) = u_0(x)
\end{cases}
\]
**Transport equation**

The $u$-equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

$$r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}$$

$$\left\{ \begin{align*}
    u_t + \frac{1}{2}r(x, t)u_x &= 0 \\
    u(x, 0) &= u_0(x)
\end{align*} \right.$$ 

Characteristic curves:

$$\frac{dx(t)}{dt} = \frac{1}{2}r(x(t), t)$$
Construction of performance surface $u(x, t)$ using characteristics

\[ \frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t) \]

Performance datum $u(x, 0)$
Construction of characteristics

\[
\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)
\]

Performance datum \( u(x, 0) \)

Characteristic curves
Propagation of performance datum along characteristics
Propagation of performance datum along characteristics
Performance surface $u(x, t)$
Two related pdes

- Fast diffusion equation for risk tolerance

\[
\begin{cases}
    r_t + \frac{1}{2}r^2 r_{xx} = 0 \\
    r(x, 0) = r_0(x)
\end{cases}
\]  
\text{(FDE)}

Conductivity : \( r^2 \)

- Porous medium equation for risk aversion

\[
\gamma(x, t) = \frac{1}{r(x, t)}
\]

\[
\begin{cases}
    \gamma_t = \frac{1}{2} \left( \frac{1}{\gamma} \right)_{xx} \\
    \gamma(x, 0) = \frac{1}{r_0(x)}
\end{cases}
\]  
\text{(PME)}

Pressure : \( r^2 \) \quad \text{and} \quad (\text{PME}) \text{ exponent:} \quad m = -1
Difficulties

- **Differential input equation:** \( u_t u_{xx} = \frac{1}{2} u_x^2 \)
  
  Inverse problem and fully nonlinear

- **Transport equation:** \( u_t + \frac{1}{2} r(x, t) u_x = 0 \)
  
  Shocks, solutions past singularities

- **Fast diffusion equation:** \( r_t + \frac{1}{2} r^2 r_{xx} = 0 \)
  
  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- **Porous medium equation:** \( \gamma_t = \frac{1}{2} (\frac{1}{\gamma})_{xx} \)
  
  Majority of results for (PME), \( \gamma_t = (\gamma^m)_{xx} \), are for \( m > 1 \), partial results for \(-1 < m < 0\)
A rich class of risk tolerance differential inputs

- Additively separable risk tolerance

\[ r^2(x, t; \alpha, \beta) = m(x; \alpha, \beta) + n(t; \alpha, \beta) \]

Example

\[ m(x; \alpha, \beta) = \alpha x^2 \quad n(x; \alpha, \beta) = \beta e^{-\alpha t} \]

\[ r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0 \]

(Very) special cases

\[ r(x, t; 0, \beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x, t) = -e^{\frac{-x}{\sqrt{\beta}} + \frac{t}{2}} \]

\[ r(x, t; 1, 0) = |x| \quad \longrightarrow \quad u(x, t) = \log x - \frac{t}{2} \]

\[ r(x, t; \alpha, 0) = \sqrt{\alpha} \ |x| \quad \longrightarrow \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)t}}, \quad \gamma = \frac{\sqrt{\alpha-1}}{\sqrt{\alpha}} \]
Multiplicatively separable risk tolerance

\[ r(x, t; \alpha, \beta) = m(x; \alpha)n(t; \beta) \]

Example

\[ m(x; \alpha) = \varphi(\Phi^{-1}(x; \alpha)) \quad n(t; \beta) = \frac{1}{\sqrt{t + \beta}}, \quad \beta > 0 \]

\[ \Phi(x; \alpha) = \int_{\alpha}^{x} e^{z^2/2} \, dz \quad \varphi = \Phi' \]

\[ r(x, t; \alpha, \beta) = \varphi(\Phi^{-1}(x; \alpha)) \]

(Very) special cases

\[ m(x; \alpha) = \alpha, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = -e^{-\frac{x}{\alpha} + \frac{t}{2}} \]

\[ m(x; \alpha) = x, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = \log x - \frac{t}{2} \]

\[ m(x; \alpha) = \alpha x, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = \frac{1}{\gamma} x^{\gamma} e^{-\frac{\gamma}{2(1-\gamma)} t}, \quad \gamma = \frac{\alpha - 1}{\alpha} \]
Summary on differential input

- Key state variables: wealth and risk tolerance

- Risk tolerance solves a fast diffusion equation posed inversely in time

\[
\begin{aligned}
rt + \frac{1}{2}r^2 r_{xx} &= 0 \\
ru_{x}(x, 0) &= -\frac{u_0'(x)}{u_0'(x)}
\end{aligned}
\]

- Transport equation

\[
\begin{aligned}
Ut + \frac{1}{2}r(x, t)u_x &= 0 \\
u(x, 0) &= u_0(x)
\end{aligned}
\]

Forward performance process constructed by compiling differential input and stochastic market input
Stochastic market input
Investment universe

Riskless and risky securities

- \((\Omega, \mathcal{F}, \mathbb{P})\) ; \(W = (W^1, \ldots, W^d)\) standard Brownian Motion

- Traded securities

\[
1 \leq i \leq k \quad \begin{cases} 
    dS^i_t = S^i_t(\mu^i_t dt + \sigma^i_t \cdot dW_t) , & S^i_0 > 0 \\
    dB_t = r_t B_t dt , & B_0 = 1 
\end{cases}
\]

\(\mu_t, r_t \in \mathbb{R}, \sigma^i_t \in \mathbb{R}^d\) bounded and \(\mathcal{F}_t\)-measurable stochastic processes

- Postulate existence of a \(\mathcal{F}_t\)-measurable stochastic process \(\lambda_t \in \mathbb{R}^d\) satisfying

\[
\mu_t - r_t \mathbf{1} = \sigma^T_t \lambda_t
\]
Investment universe

- Self-financing investment strategies \( \pi_t^0, \pi_t^i, \ i = 1, \ldots, k \)

- Present value of this allocation

\[
X_t = \sum_{i=0}^{k} \pi_t^i
\]

\[
dX_t = \sum_{i=0}^{k} \pi_t^i (\mu_t^i - r_t) dt + \sum_{i=0}^{k} \pi_t^i \sigma_t^i \cdot dW_t
\]

\[
= \sigma_t \pi_t \cdot (\lambda_t dt + dW_t)
\]

\[
\pi_t = (\pi_t^1, \ldots, \pi_t^k), \quad \mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t
\]
Market input processes

\((\sigma_t, \lambda_t)\) and \((Y_t, Z_t, A_t)\)

These \(\mathcal{F}_t\)-mble processes do not depend on the investor’s differential input. They reflect and represent, respectively

\((\lambda_t, \sigma_t)\) : dynamics of traded securities

\(Y_t\) : benchmark numeraire

\(Z_t\) : market view away from market equilibrium feasibility and trading constraints

\(A_t\) : subordination
The processes \((Y_t, Z_t, A_t)\)

- Benchmark and/or numeraire

A “replicable” process \(Y_t\) satisfying

\[
\begin{cases}
  dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\
  Y_0 = 1
\end{cases}
\]

\(\delta_t \in \mathcal{F}_t\), \(\sigma_t \sigma_t^+ \delta_t = \delta_t\)

\(\sigma_t^+\) : Moore-Penrose matrix inverse
Market input processes

- **Market views, feasibility and trading constraints**

  An exponential martingale \( Z_t \) satisfying

  \[
  \begin{aligned}
  dZ_t &= Z_t \phi_t \cdot dW_t \\
  Z_0 &= 1 , \quad \phi_t \in \mathcal{F}_t
  \end{aligned}
  \]

- **Subordination**

  A non-decreasing process \( A_t \) solving

  \[
  \begin{aligned}
  dA_t &= \left| \delta_t - \sigma_t \sigma_t^+ (\lambda_t + \phi_t) \right|^2 dt \\
  A_0 &= 0
  \end{aligned}
  \]
Forward performance process
Optimal asset allocation
Forward performance process

Stochastic input: \((Y_t, Z_t, A_t)\)  
Differential input: \(u(x, t)\)

Benchmark

\[ u_t u_{xx} = \frac{1}{2} u_x^2 \]
\[ u(x, 0) = u_0(x) \]

Time change \(A_t\)  
Market view \(Z_t\)

\[ U_t(x) = u\left(\frac{x}{Y_t}, A_t\right)Z_t \]
Forward performance process

Stochastic market input

\[ \lambda_t, \sigma_t \]

\[ \downarrow \]

benchmark, views

subordination

\( (Y_t, Z_t, A_t) \)

Differential input

\[ x, r_0(x) = -\frac{u'(x)}{u''(x)} \]

\[ \downarrow \]

\[ r_t + \frac{1}{2} \sigma^2 r_{xx} = 0 \] (FDE)

\[ u_t + \frac{1}{2} \sigma^2 u_x = 0 \] (TE)

\[ u(x, t) \]

\[ U_t(x) = u(\frac{x}{A_t}, Y_t) Z_t \]

Model independent construction!
What is the optimal allocation?

Optimal portfolio processes

\[ \pi_t = (\pi_t^0, \pi_t^1, \ldots, \pi_t^k) \]

can be directly and explicitly characterized

along with the construction of the forward performance!
The structure of optimal portfolios

\[ dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t) \]

**Stochastic input**
- Market
  - \((Y_t, Z_t, A_t)\)
  - \(\lambda_t, \sigma_t, \delta_t, \phi_t\)

**Differential input**
- Individual
  - wealth \(x\)
  - risk tolerance \(r(x, t)\)

\[ \frac{1}{Y_t} \pi_t^* \text{ is a linear combination of (benchmarked) optimal wealth and subordinated (benchmarked) risk tolerance} \]
Optimal asset allocation

- Let $X_t^*$ be the optimal wealth, $Y_t$ the benchmark and $A_t$ the subordination processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\hat{X}_t^* \triangleq \frac{X_t^*}{Y_t} \quad \text{and} \quad \hat{r}_t^* \triangleq r(\hat{X}_t^*, A_t)$$

Optimal (benchmarking) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = \sigma_t^+ (\lambda_t + \phi_t - \delta_t) \hat{r}_t^* + \delta_t \hat{X}_t^*$$
Stochastic evolution of wealth-risk tolerance
A system of SDEs at the optimum

\[ \hat{X}_t^* = \frac{X_t}{Y_t} \quad \text{and} \quad \hat{r}_t^* = r(\hat{X}_t^*, A_t) \]

\[
\begin{cases}
    d\hat{X}_t^* = \hat{r}_t^*(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\
    d\hat{r}_t^* = r_x(\hat{X}_t^*, A_t) d\hat{X}_t^*
\end{cases}
\]

- **Separability** of wealth dynamics in terms of risk tolerance and market input
- **Sensitivity** of risk tolerance in terms of its spatial gradient and changes in optimal wealth

Universal representation, no Markovian assumptions
**Wealth-Risk tolerance**

Optimal wealth-risk tolerance \((\hat{X}_t^*, \hat{r}_t^*)\) system of SDEs in original market configuration

\[
\begin{align*}
    d\hat{X}_t^* &= \hat{r}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\
    dr_t^* &= r_x(\hat{X}_t^*, A_t) d\hat{X}_t^*
\end{align*}
\]

- change of measure: historical → benchmarked
- change of time: Levy’s theorem
Wealth-Risk tolerance

Optimal wealth-risk tolerance \((x^1_t, x^2_t)\) system of SDEs

in canonical market configuration

\[
\begin{align*}
x^1_t &= \left( \frac{X^*_t}{Y_t} \right) A_t^{(-1)} \\
x^2_t &= r \left( \frac{X^*_t}{Y_t}, A_t \right) A_t^{(-1)}
\end{align*}
\]

\[
\langle M_t \rangle = A_t \\
w_t = M_{A^{(-1)}}
\]

\[
\begin{cases}
dx^1_t = x^2_t \, dw_t \\
dx^2_t = r_x(x^1_t, t) x^2_t \, dw_t \\
x^1_0 = \frac{x}{y}, \quad x^2_0 = r_x(\frac{x}{y}, 0)
\end{cases}
\]
Analytic solution of the SDE system

\[
\begin{cases}
    dx_1^t = x_2^t \, dw_t \\
    dx_2^t = r_x(x_1^t, t) x_2^t \, dw_t
\end{cases}
\]

- Define the budget capacity function $h(x, t)$ via

\[
x = \int x \, \frac{du}{r(u, t)} = \int x \, \gamma(u, t) \, du
\]

$x$ : related to symmetry properties of risk tolerance
Analytic solutions

The budget capacity function $h$ solves the (inverse) heat equation

\[
\begin{cases}
ht + \frac{1}{2}h_{xx} - \frac{1}{2}r_x(x, t)h_x = 0 \\
h(x, 0) = h_0(x), \quad x = \int_x^{h_0(x)} \frac{du}{r(u, 0)}
\end{cases}
\]

Solution of the SDE system

\[
\begin{align*}
x_1^t &= h(z_t, t) \\
x_2^t &= h_z(z_t, t)
\end{align*}
\]

\[
z_t = h_0^{-1}(x) - \int_0^t \frac{1}{2}r_x(x, s)ds + w_t
\]

Using equivalent measure transformations and time change we recover the original pair of optimal (benchmarked) wealth and (benchmark) risk tolerance.
Forward performance measurement

**Market**
- Benchmark, views, constraints
- Market input processes
- Subordination

**Investor**
- Wealth, risk tolerance
- Fast diffusion eqn
- Transport eqn

Forward evolution
\[ Y_t, Z_t, A_t \]
\[ x, r(x, t), u(x, t) \]

Optimal performance and optimal portfolios

Wealth-Risk tolerance SDE system

- Heat eqn
- Fast diffusion eqn

Universal analytic solutions
Forward performance measurement

Market

Benchmark, views, constraints

Market input processes

Subordination

Investor

Wealth, risk tolerance

Fast diffusion eqn

Transport eqn

Forward evolution

\[ Y_t, Z_t, A_t \]

\[ x, r(x, t), u(x, t) \]

Optimal performance and optimal portfolios

Wealth-Risk tolerance SDE system

Heat eqn

Fast diffusion eqn

Universal analytic solutions
An example
Forward exponential performance

Objective: Find an \( \mathcal{F}_t \)-adapted process \( U_t(x) \) such that

\[
\begin{align*}
U_0(x) &= -e^{-x} \\
E_{\mathbb{P}} \left( U_s (X^\pi_s) \mid \mathcal{F}_t \right) &\le U_t (X^\pi_t) \\
E_{\mathbb{P}} \left( U_s (X^\pi^*_s) \mid \mathcal{F}_t \right) &= U_t (X^\pi^*_t), \quad s \ge t
\end{align*}
\]

Solution

- Differential input

\[
u(x, t) = -e^{-x + \frac{1}{2} t}
\]
Forward exponential performance (continued)

- **Stochastic market input** \((Y, Z, A)\)

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\
\quad \quad Y_0 = y > 0, \\
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad dZ_t = Z_t \phi_t \cdot dW_t \\
\quad \quad Z_0 = 1, \\
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad dA_t = \left| \delta_t - \sigma_t \sigma_t^+ (\lambda_t + \phi_t) \right|^2 dt \\
\quad \quad A_0 = 0, \\
\end{array} \right.
\end{align*}
\]
Solutions

• Forward performance

\[ U_t(x) = -\exp \left( -\frac{x}{Y_t} + \frac{1}{2} \int_0^t |\delta_s - \sigma_s \sigma_t^+ (\lambda_s + \phi_s)|^2 ds \right. \]
\[ \left. - \frac{1}{2} \int_0^t |\phi_s|^2 ds + \int_0^t \phi_s \cdot dW_t \right) \]

• Feedback portfolio control process

\[ \pi_t^* = Y_t \sigma_t^+ (\lambda_t + \phi_t - \delta_t) + X_t^* \sigma_t^+ \delta_t \]

• Optimal wealth process

\[ dX_t^* = \left( Y_t \left( \sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t \right) + X_t^* \delta_t \right) \cdot (\lambda_t dt + dW_t) \]

• Optimal performance

\[ dU_t(X_t^*) = U_t(X_t^*) \left( \sigma_t \sigma_t^+ (\delta_t - \lambda_t) + \left( I - \sigma_t \sigma_t^+ \right) \phi_t \right) \cdot dW_t \]
Examples

Case 1: **No benchmark and ‘no’ views** \( \delta = \phi = 0 \)

Then, \( Y_t = y \), for \( t \geq 0 \)

- **Forward performance process**

  \[
  U_t(x) = -\exp \left( -\frac{x}{y} + \int_0^t \frac{1}{2} \left| \sigma_s \sigma_s^+ \lambda_s \right|^2 ds \right)
  \]

  Note that even in this simple case, the solution is equal to the classical exponential “utility” only at \( t = 0 \)

- **Optimal discounted wealth and optimal asset allocation**

  \[
  X_t^* = x + \int_0^t y \left( \sigma_s \sigma_s^+ \lambda_s \right) \cdot (\lambda_s ds + dW_s)
  \]

  and

  \[
  \pi_t^* = y \sigma_t^+ \lambda_t
  \]

  Observe that \( \pi^* \) is independent of the initial wealth \( x \)
Case 1: **No benchmark and ‘no’ views** $\delta = \phi = 0$ (continued)

- Optimal performance

\[
U_t (X_t^*) = - \exp \left( - \frac{x}{y} - \int_0^t \frac{1}{2} \left| \sigma_s \sigma^+_s \lambda_s \right|^2 ds - \int_0^t \sigma_s \sigma^+_s \lambda_s \cdot dW_s \right)
\]

- Total amount allocated in the risky assets

\[
1 \cdot \pi_t^* = 1 \cdot y \sigma^+_t \lambda_t
\]

- Amount invested in the riskless asset

\[
\pi_t^{0,*} = X_t^* - 1 \cdot y \sigma^+_t \lambda_t
\]

Such an allocation is rather **conservative** and is often viewed as an argument **against** the classical exponential criteria.
Case 2: **No benchmark and risk neutralization** \( \delta = 0 \) and \( \lambda + \phi = 0 \)

Then, \( \mathcal{E}_t = 1 \), \( Y_t = y > 0 \) and

\[
Z_t = e^{-\int_0^t \frac{1}{2}|\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s}
\]

- **Forward exponential performance process**

\[
U_t(x) = -\exp \left( -\frac{x}{y} - \frac{1}{2} \int_0^t |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s \right)
\]

- **Optimal discounted wealth**

\[
X_t^* = x
\]
Case 2: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \phi = 0$

(continued)

- Optimal allocations
  $$\pi_t^* = 0 \quad \text{and} \quad \pi_0^*, t = X_t^* = x$$

- Optimal exponential performance
  $$U_t(X_t^*) = U_t(x)$$

\[
= -\exp \left( -\frac{x}{y} - \frac{1}{2} \int_0^t |\lambda_s|^2 \, ds - \int_0^t \lambda_s \cdot dW_s \right)
\]

It is important to notice that, for all trading times, the optimal allocation consists of putting zero into the risky assets and, therefore, investing the entire wealth into the riskless asset. Such a solution seems to capture quite accurately the strategy of a derivatives trader for whom the underlying objective is to hedge as opposed to the asset manager whose objective is to invest
Case 3: **Following the benchmark** \( \delta = \lambda + \phi \) with \( \lambda + \phi \neq 0 \)

Then \( \delta = \sigma \sigma^+ (\lambda + \phi) \) and \( Z_t = \exp \left( - \int_0^t \frac{1}{2} |\phi_s|^2 \, ds + \int_0^t \phi_s \cdot dW_s \right) \)

- **Forward exponential performance process**

\[
U_t(x) = - \exp \left( - \frac{x}{Y_t} - \int_0^t \frac{1}{2} |\phi_s|^2 \, ds + \int_0^t \phi_s \cdot dW_s \right)
\]

- **Optimal wealth**

\[
X_t^* = x \mathcal{E}_t
\]

- **Returns of wealth and of benchmark**

\[
\frac{dX_t^*}{X_t^*} = \frac{dY_t}{Y_t}
\]
Case 3: **Following the benchmark** \( \delta = \lambda + \phi \) with \( \lambda + \phi \neq 0 \)

(continued)

- **Optimal allocation**

  \[
  \pi_t^* = \phi_s X_t^* \sigma_t^+ \delta_t
  \]

- **Optimal exponential performance**

  \[
  U_t (X_t^*) = - \exp \left( - \frac{x}{Y_t} - \int_0^t \frac{1}{2} |\phi_s|^2 \, ds + \int_0^t \phi_s \cdot dW_s \right)
  \]

Observe that, contrary to what we have observed in traditional (backward) exponential utility problems, the optimal portfolio is a **linear functional of the wealth** and not independent of it.
Case 4: **Generating arbitrary portfolio allocations**

- Assume that $1 \cdot \sigma_t^+(\lambda_t + \phi_t) = 1$. Then

$$1 \cdot \pi_t^* = X_t^* \quad \text{and} \quad \pi_t^{0,*} = 0$$

Hence, the optimal allocation $\pi^*$ puts **zero** amount in the riskless asset and invests **all** wealth in the risky assets, according to the weights specified by the vector $\sigma^+(\lambda + \phi)$.
Case 4: Generating arbitrary portfolio allocations (continued)

- Note, also, that for an arbitrary vector \( \nu_t \) with \( \mathbf{1} \cdot \sigma_t^+ \nu_t \neq 0 \), the vector

\[
\phi_t = \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{1 \cdot \sigma_t^+ \nu_t} \nu_t
\]

satisfies the above constraint since

\[
1 \cdot \sigma_t^+ \left( \lambda_t + \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{1 \cdot \sigma_t^+ \nu_t} \nu_t \right) = 1
\]

Can we generate optimal portfolios that allocate arbitrary, but constant, fractions of wealth to the different accounts?

The answer is affirmative. Indeed, for \( p \in \mathcal{R} \), set,

\[
\mathbf{1} \cdot \sigma_t^+ (\lambda_t + \phi_t) = p
\]

Then, the total investment in the risky assets and the allocation in the riskless bond are

\[
\mathbf{1} \cdot \pi_t^* = pX_t^* \quad \text{and} \quad \pi_t^{0,*} = (1 - p) X_t^*
\]