Dynamic utility pricing systems

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Towards a constitutive analogue of the Black and Scholes theory in incomplete markets
Modelling, optimal behavior, valuation and risk management

- Market prices of underlying securities
- Risk preference formulation
- Specification of admissible strategies
- Construction of optimal strategies
- Valuation

In complete markets only the first and last step are important
Fundamental elements of an indifference pricing system

Absence of static arbitrage

- Numeraire independence
- Monotonicity, scaling with respect to payoffs
- Monotonicity, robustness with respect to risk aversion
- Translation invariance with respect to replicable risks
- Risk quantification and monitoring
- Zero indifference value for residual risks
Outline

Motivational examples from indifference valuation

- Numeraire invariance
  - Stochastic risk tolerance
- Semigroup property of prices
  - Invariance of optimal utilities across trading horizons

Stochastic utilities

- Normalization
  - Evolution
    - Backward
    - Forward
References

• “Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model”
  Indifference Pricing, PUP (2005, Musiela-Z.)

• “Investment and valuation under backward and forward dynamic utilities in a stochastic factor model”
  Preprint (2005, Musiela-Z.)

• “Numeraire consistency, stochastic risk preferences and indifference valuation”
Ingredients of a utility-based pricing system

• Semigroup property of prices

\[ \nu_t(C_T; \gamma(\omega)) = \nu_t(\nu_s(C_T; \gamma(\omega)); \gamma(\omega)) \]

• Numeraire invariance

\[ \frac{\nu^Y_t(C_T; \gamma^Y(\omega))}{Y_t} = \nu^Z_t \left( \frac{C_T}{Y_T}; \gamma^Z(\omega) \right) \]
No arbitrage

The term structure of risk preferences

Stochastic utilities
The term structure of risk preferences
Fundamental questions

• What is the proper specification of the investors’ risk preferences?

• Are risk preferences static or dynamic?

• Are they affected by the market environment and the trading horizon?

• Are there endogenous structural conditions on risk preferences?

• How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?
Requirements for a well posed indifference pricing system

(work in progress MZ, ZZ)

Risk preferences need to satisfy structural conditions across units, trading horizons and maturities

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Self-generating stochastic utilities

Martingality of risk tolerance process is one of the requirements we observed so far
Self-generating stochastic utilities

Utility process:

\[ U_s \quad U_t \quad U_T \quad U_t \in \mathcal{F}_t \]

Value function process:

\[ V_s \quad U_t \quad U_T \quad V_t \in \mathcal{F}_t \]

\[ V_s(X_s; T) = \sup_{\mathcal{A}} E_{\mathbb{P}}[U_t(X_t; T)/\mathcal{F}_s] \]

Requirement for correct pricing

\[ V_s = U_s \quad V_t = U_t \quad V_t = U_T \]

Alignment of current utility with its associated value function
Self-generation of dynamic utility

- No static arbitrage
- Consistency of prices across units
- Semigroup property of prices
- Invariance, under optimal investment behavior, with respect to trading horizons and maturities
Models of the term structure of risk preferences

- Normalization point
- Evolution equation

Questions

- Are dynamic risk preferences and market behavior interlinked?
- How is the utility evolving w.r.t. time?

Answers

- Preferences and market cannot be defined in isolation
- Utilities are defined backward and forward in time
Backward stochastic utilities

- Traded asset: $S_t$, $t \in [0, T]$, Non-traded asset: $Y_t$, $t \in [0, T]

Filtrations: $\mathcal{F}_t^S$, $\mathcal{F}_t^Y$, $\mathcal{F}_t^{(S,Y)}$

- Normalization point: $T > 0$

- The backward, normalized at $T > 0$, dynamic utility $U^B_t(x; T)$ is defined as an $\mathcal{F}_t^{(S,Y)}$-measurable process solving

$$U^B_S(X_s; T) = \sup_{\alpha} E_{\mathbb{P}} \left( U^B_t(X_t; T) | \mathcal{F}_s^{(S,Y)} \right)$$

$$X_t = X_s + \int_s^t \alpha_u dS_u$$

$$U^B(x, T; T) = U(x; \gamma_T) ; \quad U \text{ given}$$
Backward stochastic utilities

\[ U_s^B(X_s; T) = \sup_{\mathcal{A}} E[U_t^B(X_t; T)/\mathcal{F}_s] = \sup_{\mathcal{A}} E[U^B(X_T; T)/\mathcal{F}_s] = V_s(X_s; T) \]

- Self-generation and invariance w.r.t. trading horizons
- Backward utility is, essentially, the traditional value function
- Backward utility and market behavior interlinked
Complete market and stochastic risk tolerance

Example

\[ U_T^B = -e^{-\gamma T X_T} \]

\[ U_s^B(X_s; T) = \sup_{A} E_{\mathbb{P}} \left[ U_t^B(X_t; T)/\mathcal{F}_s \right] = -e^{-\frac{X_s}{\delta_s}}\tilde{H}_s \]

\[ \delta_s = E_{\mathbb{Q}} \left( \frac{1}{\gamma T}/\mathcal{F}_s \right) \quad \tilde{H}_s = E_{\tilde{\mathbb{Q}}} \left[ \int_s^T \frac{1}{2} \lambda_u^2 du/\mathcal{F}_s \right] \]

\[ S_s = E_{\mathbb{Q}}(S_T/\mathcal{F}_s) \quad \frac{1}{\delta_s} S_s = E_{\tilde{\mathbb{Q}}} \left( \frac{1}{\delta_T} S_t/\mathcal{F}_s \right) \]
Forward stochastic utility (MZ 2005)

- How do we decide today about our future risk attitude?
- How do we valuate, via utility, claims of arbitrary maturities?

\[
\begin{array}{c}
\text{fixed horizon} \\
T \\
\hline
\end{array}
\]

\[
t \quad C_s, \ldots, C_T
\]

\[
\begin{array}{c}
\text{new claim arrives} \\
\uparrow
\end{array}
\]

\[
\text{valuation already done w.r.t. normalization point } T
\]

\[
T
\]

\[
T' \\
\hline
\]

longer maturity
Forward stochastic utility (con’)

- Normalization point $s > 0$

- Dynamic utility is specified going forward in time, $t \geq s$

- No need to have a fixed trading horizon
Forward stochastic utility

- Normalization point: $s > 0$

- $U_t^F(x; s), s > 0, t \geq s$ is defined as an $\mathcal{F}^{(S,Y)}_t$-measurable process solving

\[
U_t^F(X_t; s) = \sup_{\alpha} E_{\mathbb{P}}(U^{F}_{t'}(X_t; s)|\mathcal{F}^{(S,Y)}_t)
\]

\[
X_{t'} = X_t + \int_t^{t'} \alpha_u dS_u
\]

$U_s^F(x) = U(x; \omega) ; \ U \ \text{given}$
Forward stochastic utilities

\[ U^F_s(x; s) \]

utility generation forward in time

\[ U^F_t(X_t; s) = \sup_{A} E_{\mathbb{P}} [U^F_{t'}(X_{t'}; s)/\mathcal{F}_s] \]

\[ = V^F_t(X_t; s) \]

Self-generation and invariance w.r.t. trading horizon
Complete market case and stochastic Sharpe ratio

Example

\[ U_s^F (x; s) = -e^{-\gamma x} + \int_s^t \frac{1}{2} \lambda_u^2 \, du \]

Self-generation

\[ -e^{-\gamma x} + \int_s^t \frac{1}{2} \lambda_u^2 \, du = \sup_{\mathcal{A}} \mathbb{E}_\mathbb{P} \left[ -e^{-\gamma X_{t'}} + \int_t^{t'} \frac{1}{2} \lambda_u^2 \, du / \mathcal{F}_t \right] \]
Forward versus backward utilities

- **Backward** stochastic utilities aggregate market information while **forward** stochastic utilities use information revealed *dynamically* by the market.

- **Forward** indifference prices do **not** depend on the preference normalization point.

- Forward indifference prices are represented in a *more intuitive* manner.
Open questions

- When do backward stochastic utilities exist? Are they unique? Are they robust w.r.t. terminal risk preference specification? (relatively easy)

- When do forward stochastic utilities exist? Are they unique? Are they robust w.r.t. initial risk preference specification? (very hard)

- What is the term structure of the risk tolerance process in backward and forward setting?

- How is indifference valuation built in terms of backward and forward preferences?

- Are backward and forward indifference prices equal?

- How do they depend on the risk preference normalization point?
Results to date

Backward and forward stochastic utilities and their associated prices and hedges

• Binomial model
• Diffusion model
• Stochastic risk tolerance

Exponential preferences

Power preferences (partial results)
Utility-based backward and forward pricing systems for stochastic volatility models
The diffusion case

• Market dynamics

\[ dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW_s^1 \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]
\[ \rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

• Minimal relative entropy measure: \( \mathbb{Q} \)

\[ \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( - \int_0^T \lambda_s \, dW_s^1 - \int_0^T \lambda_s^1 \, dW_s^{1,\perp} - \frac{1}{2} \int_0^T (\lambda_s^2 + (\lambda_s^1)^2) \, ds \right) \]
\[ \lambda_s^\perp = \lambda_s^\perp(Y_s, s); \lambda^\perp(y, t) \sim \text{gradient to the sln of a quasilinear pde} \]

(Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)
**Backward stochastic utilities**

For $T > 0$ the process $\{U^B_t(x;T) : 0 \leq t \leq T\}$ defined, for $x \in \mathbb{R}$, by

$$U^B_t(x;T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x - H_T(Q(\cdot|\mathcal{F}_t)|P(\cdot|\mathcal{F}_t))} & \text{if } 0 \leq t < T 
\end{cases}$$

with

$$H_T(Q(\cdot|\mathcal{F}_t)|P(\cdot|\mathcal{F}_t)) = E_Q\left(\int_t^T \frac{1}{2}(\lambda_s^2 + (\lambda_s^\perp)^2) \, ds | \mathcal{F}_t\right)$$

is the, normalized at time $T$, backward stochastic exponential utility

28
Forward stochastic utilities (MZ 2005)

For $T \geq s \geq 0$ and $x \in \mathbb{R}$, the process $U^F_t(x; s) e^{\mathcal{F}^{(S,Y)}_t}$ defined by

$$U^F_t(x; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x + h_t} & \text{if } t \geq s 
\end{cases}$$

with

$$h_t = \int_s^t \frac{1}{2} \lambda^2(Y_u, u) \, du$$

is the forward, normalized at time $s$, stochastic exponential utility.
Backward and forward indifference prices

- Backward price
  \[ h_t^B(C_{\hat{T}}; T) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t, \hat{T})}(C_{\hat{T}}; T) \]
  \( \hat{T} \leq T \), \( \mathbb{Q}^{mm} \): minimal martingale measure

- Forward exponential utility
  \[ h_t^F(C_{\hat{T}}; s) = \mathcal{E}_{\mathbb{Q}^{me}}^{(t, \hat{T})}(C_{\hat{T}}) \]
  No dependence on \( s \), \( \mathbb{Q}^{me} \): minimal entropy measure
Before we “become indifferent” and quote prices, we need to understand

- how we formulate our risk appetite
- how our risk tolerance and the market environment affect each other
- how we built our dynamic utility structure

Technical challenges

- Stochastic pdes
- Forward utility might give rise to “not well-posed” problems