Portfolio choice under space-time monotone performance criteria*

M. Musiela† and T. Zariphopoulou‡

October 8, 2009

Abstract

The class of time-decreasing forward performance processes is analyzed in a portfolio choice model of Ito-type asset dynamics. The associated optimal wealth and portfolio processes are explicitly constructed and their probabilistic properties discussed. These formulae are, in turn, used in analyzing how the investor's preferences can be calibrated to the market, given his desired investment targets.

1 Introduction

This paper is a contribution to portfolio management from the perspective of investor preferences and, hence, in its spirit is related to the classical expected utility maximization problem introduced by Merton ([8]). Therein, one first chooses an investment horizon and assigns a utility function at the end of it and, in turn, seeks an investment strategy which delivers the maximal expected (indirect) utility of terminal wealth. Recently, the authors proposed an alternative approach to optimal portfolio choice which is based on the so-called forward performance criterion (see, among others, [10] and [9]). In this approach, the investor does not choose her risk preferences at a single point in time, as it is the case in the Merton model, but has the flexibility to revise them dynamically.

*This work was presented at the 4^{th} World Congress of BFS (2006) and at meetings and workshops at, among others, U.C., Santa Barbara, U.M., College Park, Columbia University, MIT (2006), Oberwolfach, Stanford University, AMaMeF and QMF (2007), and Oberwolfach, University of Kostanz and RICAM (2008). The authors would like to thank the participants for fruitful suggestions and, especially, P. Carr and M. Yor. They, also, thank two anonymous referees for their valuable comments. The first version of this paper (September 2007) was titled *"Implied preferences and bespoke portfolios"*.

†BNP Paribas, London; marek.musiela@bnpparibas.com

‡University of Oxford, Oxford-Man Institute and Mathematical Institute, and The University of Texas at Austin, Department of Mathematics and Department of IROM, McCombs School of Business; Thaleia.Zariphopoulou@oxford-man.ox.ac.uk. The second author acknowledges partial support from the National Science Foundation (NSF grants: DMS-FRG-0456118 and DMS-RTG-0636586).

Herein, we focus on a specific case of a forward performance criterion, originally introduced in [12]. This criterion is a composition of deterministic and stochastic inputs. The deterministic input corresponds to the investor's preferences, and alternatively, to her tolerance towards risk. It is investor specific, represented by a function $u(x,t)$, which is increasing and concave in x, and decreasing in t. The stochastic input, however, is universal for all investors and is given by $A_t = \int_0^t |\lambda_s|^2 \, ds$, $t \geq 0$, with λ_t being the Sharpe ratio of the available for trading securities. The performance criterion is, then, given by the process $U_t(x) = u(x, A_t)$, $t \geq 0$. Because of its form and the properties of the involved inputs, the performance process is monotone in wealth and time.

Our contribution is threefold. Firstly, we provide a general characterization of the differential input function, $u(x,t)$. A space-time harmonic function plays a pivotal role in achieving this. This function is fully characterized by a positive measure which, in turn, becomes the underlying element in the specification of all quantities of interest. An important ingredient is the support of this measure, as it directly affects the domain of the differential input. We provide a detailed study of this interplay.

The second contribution is the explicit construction of the optimal investment strategy and the associated optimal wealth. The specification of these processes is rather general as it does not rely on any Markovian assumptions on asset dynamics or on any specific structure of the investor's input. To our knowledge, this is one of the very rare cases in which such explicit formulae can be derived in a model as general as the one considered herein.

The third contribution is the initiation of a study on how we can learn about the investor's risk preferences from his investment goals. For example, the investor may want to specify the average level of wealth he could generate in future times, in the particular market he chooses. This information is, then, used to deduce his preferences which are consistent with this investment target. Such inference problems are, in general, very hard to solve due to lacking closed form formulae, a difficulty that is surpassed herein due to the availability of explicit solutions. It is important to notice that the assessment of market movements is implicitly embedded in the investor's desired investment goals. In many aspects, this approach can be compared with the calibration of derivative pricing models. Indeed, therein, one also needs to make a statement about the market under the historical measure. Then, assuming no arbitrage, derivatives are valued under the risk neutral measure, with the valuation requiring calibration of the model to the observable market prices of the relevant assets. There is, however, an important difference between the derivative pricing and the portfolio selection problem. In the latter, we cannot rely on the market to give information about the investor's individual preferences. However, we show how to acquire information about them by asking the investor to specify the desirable properties of the wealth process she wishes to generate.

The criterion studied in this paper does not allow for arbitrary stochastic evolution of preferences as the performance process is monotonically decreasing and, hence, its quadratic variation is equal to zero. To incorporate more flexibility, one needs to work with selection criteria in their full generality. Preliminary results in this direction can be found in [11].

The paper is organized as follows. The model and the general portfolio selection criteria are defined in the next section. In section 3, we present the monotone performance criterion and provide explicit solutions for the associated optimal wealth and portfolio processes. In section 4, we provide a detailed construction of the differential input via the associated space-time harmonic function. In section 5, we analyze the case of deterministic market price of risk and discuss the distributional properties of the optimal wealth. We finish with discussing how the investor specific input can be inferred from targeted properties of her future expected wealth.

2 The model and portfolio selection criteria

The market environment consists of one riskless and k risky securities. The prices of risky securities are modelled as Ito processes. Namely, the price S^i of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$ for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d\right)$, $t \geq 0$, is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients μ_t^i and $\sigma_t^i = \left(\sigma_t^{1i}, ..., \sigma_t^{di}\right)$, $i = 1, ..., k$, $t \geq 0$, are $\mathcal{F}_t-$adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. For brevity, we write σ_t to denote the volatility matrix, i.e., the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. We may, then, alternatively write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right). \tag{1}$$

The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt,$$

with $B_0 = 1$, and for a nonnegative, $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the $k-$dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the $k-$dimensional vector with every component equal to one. The processes, μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that the volatility vectors are such that

$$\mu_t - r_t \mathbf{1} \in Lin\left(\sigma_t^T\right), \tag{2}$$

where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of σ_t^T. This implies that $\sigma_t^T \left(\sigma_t^T\right)^+ (\mu_t - r_t \mathbf{1}) = \mu_t - r_t \mathbf{1}$ and, therefore, the vector

$$\lambda_t = \left(\sigma_t^T\right)^+ (\mu_t - r_t \mathbf{1}) \tag{3}$$

is a solution to the equation $\sigma_t^T x = \mu_t - r_t \mathbf{1}$. The matrix $\left(\sigma_t^T\right)^+$ is the Moore-Penrose pseudo-inverse[1] of the matrix σ_t^T.

Occasionally, we will be referring to λ_t as the market price of risk. It easily follows that

$$\sigma_t \sigma_t^+ \lambda_t = \lambda_t \tag{4}$$

and, hence, $\lambda_t \in Lin\left(\sigma_t\right)$. We assume throughout that the process λ_t is bounded by a deterministic constant $c > 0$, i.e., for all $t \geq 0$,

$$|\lambda_t| \leq c. \tag{5}$$

Starting at $t = 0$ with an initial endowment $x \in \mathbb{R}$, the investor invests at any time $t > 0$ in the riskless and risky assets. The present value of the amounts invested are denoted by π_t^0 and π_t^i , $i = 1, ..., k$, respectively.

The present value of her investment is, then, given by $X_t^\pi = \sum_{i=0}^k \pi_t^i$, $t > 0$. We will refer to X_t^π as the discounted wealth. The investment strategies will play the role of control processes and are taken to satisfy the standard assumption of being self-financing. Using (1) we, then, deduce that the discounted wealth satisfies

$$dX_t^\pi = \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot \left(\lambda_t dt + dW_t\right) = \sigma_t \pi_t \cdot \left(\lambda_t dt + dW_t\right), \tag{6}$$

where the (column) vector, $\pi_t = \left(\pi_t^i;\ i = 1, ..., k\right)$. The set of admissible strategies, $\mathcal{A}$, is defined as

$$\mathcal{A} = \left\{ \pi : \text{ self-financing with } \pi_t \in \mathcal{F}_t \text{ and } E\left(\int_0^t \left|\sigma_s \pi_s\right|^2 ds\right) < \infty,\ t > 0 \right\}. \tag{7}$$

The problem we propose to address is that of a choice of an investment strategy from the set $\mathcal{A}$. To this aim, we introduce below a process which measures the performance of any admissible portfolio and gives us a selection criterion. Specifically, a strategy is deemed optimal if it generates a wealth process whose average performance is maintained over time. In other words, the average performance of this strategy at any future date, conditional on today's information, preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is, then, sub-optimal.

We present the definition of the forward performance next. It first appeared in [10] and is given herein for completeness. We note that this definition is slightly different than the original one, introduced by the authors in [12], in that the initial condition is not explicitly included. As the analysis in section 4 will show, not all strictly increasing and concave solutions can serve as initial conditions, even for the special classes of monotone processes we examine herein. Characterizing the set of appropriate initial conditions is a challenging question and is currently being investigated by the authors.

[1] See [13] and [7].

Definition 1 *An $\mathcal{F}_t$−adapted process $U_t(x)$ is a forward performance if for $t \geq 0$ and $x \in \mathbb{R}$:*

i) the mapping $x \to U_t(x)$ is strictly concave and increasing,

ii) for each $\pi \in \mathcal{A}$, $E\left(U_t\left(X_t^\pi\right)\right)^+ < \infty$, and

$$E\left(U_s\left(X_s^\pi\right)|\mathcal{F}_t\right) \leq U_t\left(X_t^\pi\right), \qquad s \geq t, \tag{8}$$

iii) there exists $\pi^* \in \mathcal{A}$, for which

$$E\left(U_s\left(X_s^{\pi^*}\right)|\mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \qquad s \geq t. \tag{9}$$

The intuition behind the above definition comes from the analogous martingale and supermartingale properties that the traditional maximal expected utility (value function) has (see, among others, [8], [5] and [14]). Indeed, we recall that the latter is defined in a finite trading horizon, say $[0,T]$, by

$$v(x,t;T) = \sup_{\mathcal{A}_T} E\left(V\left(X_T^\pi\right)\middle|\, \mathcal{F}_t, X_t^\pi = x\right), \tag{10}$$

with $(x,t) \in \mathbb{R}\times[0,T]$, where $\mathcal{A}_T$ is the set of admissible policies defined similarly to $\mathcal{A}$ herein and V is the investor's utility, given by an increasing, concave and smooth function. Under rather general model assumptions, the value function satifies the Dynamic Programming Principle (DDP), namely, for $0 \leq t \leq s \leq T$,

$$v(x,t;T) = \sup_{\mathcal{A}_T} E\left(v\left(X_s^\pi, s;T\right)\middle|\, \mathcal{F}_t, X_t^\pi = x\right). \tag{11}$$

One, then, sees that if the above supremum is achieved and certain integrability conditions hold, the processes $v\left(X_s^\pi, s;T\right)$ and $v\left(X_s^*, s;T\right)$ are, respectively supermartingale and martingale on $[0,T]$.

We stress that the analogous equivalence in the forward formulation of the problem has not yet been established. Specifically, one could define the forward performance process via the (forward) stochastic optimization problem

$$U_t(x) = \sup_{\mathcal{A}} E\left(U_s\left(X_s^\pi\right)\middle|\, \mathcal{F}_t, X_t^\pi = x\right),$$

for all $0 \leq t \leq s$ and for an appropriately defined initial condition. Characterizing its solutions poses a number of challenging questions, some of them being currently investigated by the authors[2]. From a different perspective, one could seek an axiomatic construction of a forward performance process. Results in this direction, as well as on the dual formulation of the problem, can be found in [19] for the exponential case (see, also, [1] for a constrained case). We refer the reader to [10] for further discussion on the forward performance and its similarities and differences with the classical value function.

[2] While preparing this revised version, the authors came across the revised version of [1] where similar questions are studied for the nonnegative wealth case.

The definition of the forward performance process requires the integrability of $(U_t(X_t^\pi))^+$. This allows us to define the conditional expectations $E(U_s(X_s^\pi)|\mathcal{F}_t)$ for $s \geq t$ and, in turn, obtain a more practical intuition for our criteria. This leads, however, to additional integrability assumptions which further constrain the class of forward solutions the investor may employ. On the other hand, from the applications perspective, this may help in the calibration process of the investor's initial risk preferences.

Alternatively, and simpler from the mathematical view point, we could replace conditions ii) and iii) above with corresponding *local* statements, as proposed next. To this end, we first relax the set of admissible strategies to

$$\mathcal{A}^l = \left\{ \pi : \text{ self-financing with } \pi_t \in \mathcal{F}_t \text{ and } \mathbb{P}\left(\int_0^t |\sigma_s \pi_s|^2 \, ds < \infty \right) = 1, \ t > 0 \right\}. \tag{12}$$

Definition 2 *An $\mathcal{F}_t-$adapted process $U_t(x)$ is a local forward performance if for $t \geq 0$ and $x \in \mathbb{R}$:*

i) the mapping $x \rightarrow U_t(x)$ is strictly concave and increasing,

ii) for each $\pi \in \mathcal{A}^l$, the process $U_t(X_t^\pi)$ is a local supermartingale, and

iii) there exists $\pi^* \in \mathcal{A}^l$ such that the process $U_t\left(X_t^{\pi^*}\right)$ is a local martingale.

Herein, we do not analyze the relaxed formulation of the problem but only present an example of a local forward performance (see Example 13).

Other modifications of the definition of the forward performance process are possible, all based on the same principle, namely, to choose an investment strategy that keeps the expected investment performance constant across time. For example, one can relax or modify the assumption on monotonicity and (strict) concavity. This is desirable, in particular, for the development of time consistent behavioral portfolio selection models. A natural modification would be to assume the existence of a reference point for the investor's wealth that defines gains and losses (see, for example, [4]). The mapping $x \rightarrow U_t(x)$ should be, then, concave for gains and convex for losses. The supermartingality condition in the above definitions would have to be replaced by a statement about the sign of the drift in the semimartingale decomposition of $U_t(X_t^\pi)$. Specifically, the drift would be negative when the wealth is above the reference point and positive when below. However, such modifications and extensions are beyond the scope of this paper and are mentioned here only to expose the flexibility of our definition.

We conclude mentioning that the classical Markowitz portfolio selection problem (see, among others [3] and [6]) could be also incorporated into our framework. Indeed, one would need to choose the mean level of wealth and find the portfolio that deviates from it the least, in the variance sense. Note that not all mean functions would be admissible, as it is already demonstrated in this paper (however, with respect to criteria not covering the case of variance). The variance-based criteria deserve a separate treatment which will be carried out in a future study.

3 Monotone performance processes and their optimal wealth and portfolio processes

We focus on the class of time-decreasing performance processes introduced by the authors in [10] (see, also [12] and [9]) and provide a full characterization of the associated optimal wealth and portfolio processes.

It was shown in [10] (see Theorems 4 and 8) that the performance process, $U_t(x)$, is constructed by compiling market related input with a deterministic function of space and time. Specifically, for $t \geq 0$, we have

$$U_t(x) = u(x, A_t), \tag{13}$$

where $u(x,t)$ is increasing and strictly concave in x, and satisfies

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}, \tag{14}$$

with A_t as in (20) below. It was also shown that the optimal wealth and the associated investment process, denoted respectively by X_t^* and π_t^*, $t \geq 0$, are constructed via an autonomous system of stochastic differential equations whose coefficients depend functionally on the spatial derivatives of u. Specifically, let $R : \mathbb{R}\times [0,+\infty) \to \mathbb{R}^+$ be defined as

$$R(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{15}$$

with u satisfying (14), and define (with slight abuse of notation) the process

$$R_t^* = R(X_t^*, A_t), \tag{16}$$

with A_t, $t \geq 0$, as in (20). Consider the system

$$\begin{cases} dX_t^* = R(X_t^*, A_t)\,\lambda_t(\lambda_t dt + dW_t) \\ \\ dR_t^* = R_x(X_t^*, A_t)\,dX_t^*, \end{cases} \tag{17}$$

and its solution (X_t^*, R_t^*), $t \geq 0$. Then, the process π_t^* defined by

$$\pi_t^* = R_t^* \sigma_t^+ \lambda_t \tag{18}$$

is optimal and generates the optimal wealth process X_t^*.

The main contribution of this section is the *explicit* construction of the optimal processes. We establish that, in analogy to the forward performance process, X_t^* and π_t^* are, also, given as a compilation of market input and deterministic functions of space and time. Namely, we show that

$$X_t^* = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right) \quad \text{and} \quad \pi_t^* = h_x\left(h^{(-1)}(X_t^*, A_t), A_t\right)\sigma_t^+\lambda_t,$$

where $h(x,t)$ is strictly increasing in x and solves the (backward) heat equation

$$h_t + \frac{1}{2} h_{xx} = 0, \tag{19}$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$. The function $h^{(-1)}$ stands for the spatial inverse of h.

The market input processes A_t and M_t, $t \geq 0$, are defined as

$$A_t = \int_0^t |\lambda_s|^2 \, ds \quad \text{and} \quad M_t = \int_0^t \lambda_s \cdot dW_s, \tag{20}$$

with λ_t as in (3).

The above formulae demonstrate that all quantities of interest can be fully specified as long as the market price of risk is chosen and the functions u and h are known. A considerable part of this paper is, thus, dedicated to the study of these functions and, especially, their representation and connection with each other. For the reader's convenience, we choose to present the detailed results separately. We do this because different cases for the domain and range of the functions u and h require appropriately modified and computationally tedious arguments which, if presented at this point, would obscure the clarity of the presentation. It is shown in section 4 (see Propositions 10, 14, 15 and 19) that there exists a one-to-one correspondence (modulo normalization constants) between increasing and strictly concave solutions to (14) with strictly increasing solutions to (19). It is, also, shown that the latter can be represented in terms of the bilateral Laplace transform of a *positive finite Borel measure*, denoted throughout by ν. This measure then emerges as the defining element in the entire analysis of the problem at hand. Its presence originates from the classical results of Widder (see Chapter XIV in [17] and Theorem 9) on the construction of positive solutions to the heat equation (19). In the investment model we consider, the solution h of (19) represents the optimal wealth which, however, might take arbitrary values. As a result, a more detailed study is required depending on the range of h.

In order to present the general ideas and provide some insights for the upcoming main theorem, we present the following representative case. We stress that the results below are not complete but are presented in this form in order to build intuition. The complete arguments are presented in Propositions 9 and 10.

To this end, we introduce the set of measures $\mathcal{B}^+(\mathbb{R})$ defined by

$$\mathcal{B}^+(\mathbb{R}) = \left\{ \nu \in \mathcal{B}(\mathbb{R}) : \ \forall B \in \mathcal{B}, \ \nu(B) \geq 0 \text{ and } \int_{\mathbb{R}} e^{yx} \nu(dy) < \infty, \ x \in \mathbb{R} \right\}. \tag{21}$$

Proposition 3 *i) Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then, the function h defined, for $(x,t) \in \mathbb{R} \times [0,+\infty)$, by*

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2} y^2 t} - 1}{y} \nu(dy) + C,$$

is a strictly increasing solution to (19).

ii) Assume that h above is of full range, for each $t \geq 0$ and let $h^{(-1)}$: $\mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined for $(x,t) \in \mathbb{R} \times [0,+\infty)$ and given by

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz,$$

is an increasing and strictly concave solution of (14).

We proceed with the main theorem in which we provide closed form expressions for the optimal wealth, the associated optimal investment strategy and the space-time monotone forward performance process. We state the result without making specific reference to the range of h, as well as the domain and range of u, as the different cases are analyzed in detail later. We, also, do not make any reference to the regularity of these functions since the required smoothness follows trivially from their representation.

We stress, however, that we introduce the integrability condition (22). This condition is stronger than the one needed for the representations of h (cf. (21)), and in turn of u, but sufficient in order to guarantee the admissibility of the candidate optimal policy (24). It may be relaxed if, for example, one chooses to work instead with local forward performance processes, introduced in Definition 2. For additional comments on condition (22) see the discussion after Example 13.

Theorem 4 *i) Let h be a strictly increasing solution to (19), for $(x,t) \in \mathbb{R}\times[0,+\infty)$, and assume that the associated measure ν satisfies*

$$\int_{\mathbb{R}} e^{yx+\frac{1}{2}y^2 t}\nu(dy) < \infty. \tag{22}$$

Let also A_t and M_t, $t \geq 0$, be as in (20) and define the processes X_t^* and π_t^* by

$$X_t^* = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right) \tag{23}$$

and

$$\pi_t^* = h_x\left(h^{(-1)}(X_t^*, A_t), A_t\right)\sigma_t^+\lambda_t, \tag{24}$$

$t \geq 0$, $x \in \mathbb{R}$ with h as above and $h^{(-1)}$ standing for its spatial inverse. Then, the portfolio π_t^* is admissible and generates X_t^*, i.e.,

$$X_t^* = x + \int_0^t \sigma_s \pi_s^* \cdot (\lambda_s ds + dW_s). \tag{25}$$

ii) Let u be the associated with h increasing and strictly concave solution to (14). Then, the process $u(X_t^*, A_t)$, $t \geq 0$, satisfies

$$du(X_t^*, A_t) = u_x(X_t^*, A_t)\sigma_t\pi_t^* \cdot dW_t, \tag{26}$$

with X_t^* and π_t^* as in (23) and (24).

iii) Let $U_t(x)$, $t \geq 0$, $x \in \mathbb{R}$ be given by

$$U_t(x) = u(x, A_t). \tag{27}$$

Then, $U_t(x)$ is a forward performance process and the processes X_t^* and π_t^* are optimal.

Proof. We provide the proof only when h is of infinite range since the cases of semi-infinite range can be worked out by analogous arguments.

As it was mentioned earlier, the representation of h is established in section 4. When h is of infinite range, it is given in Proposition 9, (cf. (39)), rewritten below for convenience, namely,

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy),$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$ (for simplicity we take $C = 0$ in (39)).

For $x \in \mathbb{R}$, A_t and M_t as in (20), we, then, define the process

$$N_t = h^{(-1)}(x, 0) + A_t + M_t,$$

where $h^{(-1)}$ is the spatial inverse of h. Applying Ito's formula to X_t^*, given in (23), and using (19) yields

$$dX_t^* = h_x(N_t, A_t)\, dN_t. \tag{28}$$

On the other hand, (23) and (24) imply

$$\pi_t^* = h_x(N_t, A_t)\, \sigma_t^+ \lambda_t,$$

$t \geq 0$, and (25) follows from the above and (4).

To establish that $\pi_t^* \in \mathcal{A}$, it suffices to show that the integrability condition in (7) is satisfied. Using that

$$h_x(x,t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2 t} \nu(dy)$$

(cf. (81)) and (23) we have

$$\left(h_x \left(h^{(-1)}(X_t^*, A_t), A_t \right) \right)^2$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(y_1 + y_2)\left(h^{(-1)}(x,0) + A_t + M_t\right) - \frac{1}{2}\left(y_1^2 + y_2^2\right) A_t} \nu(dy_1)\, \nu(dy_2).$$

From (4), Fubini's theorem and (20), we deduce

$$E\left(\int_0^t |\sigma_s \pi_s^*|^2 ds \right) = E\left(\int_0^t \left| h_x \left(h^{(-1)}(X_s^*, A_s), A_s \right) \sigma_s \sigma_s^+ \lambda_s \right|^2 ds \right)$$

$$= E\left(\int_0^t \left(h_x\left(h^{(-1)}\left(X_s^*, A_s\right), A_s\right)\right)^2 dA_s\right)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} E\left(\int_0^t e^{(y_1+y_2)\left(h^{(-1)}(x,0)+A_s+M_s\right)-\frac{1}{2}\left(y_1^2+y_2^2\right)A_s} dA_s\right)\nu\left(dy_1\right)\nu\left(dy_2\right)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} E\left(\int_0^{A_t} e^{(y_1+y_2)\left(h^{(-1)}(x,0)+s+\beta_s\right)-\frac{1}{2}\left(y_1^2+y_2^2\right)s} ds\right)\nu\left(dy_1\right)\nu\left(dy_2\right),$$

where $\beta_t = M_{A_t^{(-1)}}$ and $A_t^{(-1)}$ stands for the inverse of A_t, $t \geq 0$. Using that β_t, $t \geq 0$, is $N\left(0,t\right)$ we obtain

$$E\left(\int_0^t \left|\sigma_s \pi_s^*\right|^2 ds\right)$$

$$\leq \int_{\mathbb{R}}\int_{\mathbb{R}} E\left(\int_0^{c^2 t} e^{(y_1+y_2)\left(h^{(-1)}(x,0)+s+\beta_s\right)-\frac{1}{2}\left(y_1^2+y_2^2\right)s} ds\right)\nu\left(dy_1\right)\nu\left(dy_2\right)$$

$$= \int_0^{c^2 t}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{(y_1+y_2)\left(h^{(-1)}(x,0)+s\right)+y_1 y_2 s}\nu\left(dy_1\right)\nu\left(dy_2\right) ds$$

$$\leq \int_0^{c^2 t}\left(\int_{\mathbb{R}} e^{y\left(h^{(-1)}(x,0)+s\right)+\frac{1}{2}y^2 s}\nu\left(dy\right)\right)^2 ds,$$

and using (22) we conclude.

ii) The facts that u satisfies (14) and has the claimed monotonicity and strict concavity properties are established separately, in Proposition 10, where a detailed construction of this function is presented.

To show (26), we apply Ito's formula to $u\left(X_t^*, A_t\right)$, $t \geq 0$. To this end, using (28) yields

$$du\left(X_t^*, A_t\right) = u_x\left(X_t^*, A_t\right) dX_t^* + u_t\left(X_t^*, A_t\right) dA_t + \frac{1}{2}u_{xx}\left(X_t^*, A_t\right) d\left\langle X^*\right\rangle_t$$

$$= u_x\left(X_t^*, A_t\right) h_x\left(h^{(-1)}\left(X_t^*, A_t\right), A_t\right)\lambda_t \cdot dW_t$$

$$+ u_x\left(X_t^*, A_t\right) h_x\left(h^{(-1)}\left(X_t^*, A_t\right), A_t\right) dA_t$$

$$+ u_t\left(X_t^*, A_t\right) dA_t + \frac{1}{2}u_{xx}\left(X_t^*, A_t\right) d\left\langle X^*\right\rangle_t.$$

From (91) in the proof of Proposition 10, we deduce that

$$-\frac{u_x\left(X_t^*, A_t\right)}{u_{xx}\left(X_t^*, A_t\right)} = h_x\left(h^{(-1)}\left(X_t^*, A_t\right), A_t\right), \tag{29}$$

which combined with the above yields

$$du\left(X_t^*, A_t\right) = u_x\left(X_t^*, A_t\right) h_x\left(h^{(-1)}\left(X_t^*, A_t\right), A_t\right)\lambda_t \cdot dW_t$$

$$-\frac{\left(u_x\left(X_t^*,A_t\right)\right)^2}{u_{xx}\left(X_t^*,A_t\right)}dA_t+u_t\left(X_t^*,A_t\right)dA_t+\frac{1}{2}u_{xx}\left(X_t^*,A_t\right)d\left\langle X^*\right\rangle_t.$$

On the other hand, (28) gives

$$u_{xx}\left(X_t^*,A_t\right)d\left\langle X^*\right\rangle_t=u_{xx}\left(X_t^*,A_t\right)\left(h_x\left(h^{(-1)}\left(X_t^*,A_t\right),A_t\right)\right)^2dA_t$$

$$=\frac{\left(u_x\left(X_t^*,A_t\right)\right)^2}{u_{xx}\left(X_t^*,A_t\right)}dA_t.$$

Therefore,

$$du\left(X_t^*,A_t\right)=u_x\left(X_t^*,A_t\right)h_x\left(h^{(-1)}\left(X_t^*,A_t\right),A_t\right)\lambda_t\cdot dW_t$$

$$+\left(u_t\left(X_t^*,A_t\right)-\frac{1}{2}\frac{\left(u_x\left(X_t^*,A_t\right)\right)^2}{u_{xx}\left(X_t^*,A_t\right)}\right)dA_t,$$

and (26) follows since u satisfies (14).

iii) We need to establish that $U_t(x)$ satisfies all conditions in Definition 1. The facts that $u(x,A_t)$ is $\mathcal{F}_t-$adapted and that the mapping $x\to u(x,A_t)$ is increasing and strictly concave follow trivially from the properties of u and A_t.

To establish the integrability condition $E\left(U_t\left(X_t^\pi\right)\right)^+<\infty$, we work as follows. We first observe that the strict concavity of u together with (14) yields $u_t<0$ and, hence, $u(x,t)\le u(x,0)\le ax^++b$, for some positive constants a and b. Also, (6) implies

$$\left(X_t^\pi\right)^+\le x^++\frac{1}{2}\int_0^t\left|\sigma_s\pi_s\right|^2ds+\frac{1}{2}\int_0^t\left|\lambda_s\right|^2ds+\left|\int_0^t\sigma_s\pi_s\cdot dW_s\right|.$$

The integrability of $E\left(U_t\left(X_t^\pi\right)\right)^+$ then follows using (5) and that $\pi_t\in\mathcal{A}$.

To show (8), we observe that for $\pi_t\in\mathcal{A}$ and X_t^π as in (6), Ito's formula yields

$$du\left(X_t^\pi,A_t\right)=\left(u_x\left(X_t^\pi,A_t\right)\sigma_t\pi_t\cdot\lambda_t+u_t\left(X_t^\pi,A_t\right)\left|\lambda_t\right|^2+\frac{1}{2}u_{xx}\left(X_t^\pi,A_t\right)\left|\sigma_t\pi_t\right|^2\right)dt$$

$$+u_x\left(X_t^\pi,A_t\right)\sigma_t\pi_t\cdot dW_t$$

$$=\left(u_x\left(X_t^\pi,A_t\right)\sigma_t\pi_t\cdot\lambda_t+\frac{1}{2}\frac{\left(u_x\left(X_t^\pi,A_t\right)\right)^2}{u_{xx}\left(X_t^\pi,A_t\right)}\left|\lambda_t\right|^2+\frac{1}{2}u_{xx}\left(X_t^\pi,A_t\right)\left|\sigma_t\pi_t\right|^2\right)dt$$

$$+u_x\left(X_t^\pi,A_t\right)\sigma_t\pi_t\cdot dW_t$$

$$=\frac{1}{2}u_{xx}\left(X_t^\pi,A_t\right)\left|\sigma_t\pi_t+\frac{u_x\left(X_t^\pi,A_t\right)}{u_{xx}\left(X_t^\pi,A_t\right)}\lambda_t\right|^2dt+u_x\left(X_t^\pi,A_t\right)\sigma_t\pi_t\cdot dW_t,$$

where we used that u solves (14). Using the concavity of u we conclude.

To show (9) we use the form of the above drift, (29) and (24). ■

We remind the reader that the forward performance process in [10] is more general than the one in (13), namely, it is given by

$$U_t(x) = u\left(\frac{x}{Y_t}, \tilde{A}_t\right) Z_t, \tag{30}$$

where the processes (Y_t, Z_t) represent, respectively, a benchmark (or numeraire) and alternative market views. They solve

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \quad \text{ and } \quad dZ_t = Z_t \phi_t \cdot dW_t,$$

with $Y_0 = Z_0 = 1$ and δ_t, ϕ_t being $\mathcal{F}_t-$adapted processes, satisfying $\sigma_t \sigma_t^+ \delta_t = \delta_t$ and $\sigma_t \sigma_t^+ \phi_t = \phi_t$, $t \geq 0$. The process $\tilde{A}_t$ has a similar form to (20),

$$\tilde{A}_t = \int_0^t |\lambda_s + \phi_s - \delta_s|^2 \, ds.$$

Herein, we assume throughout $\delta_t = \phi_t = 0$, $t \geq 0$, as we focus on monotone in time forward performance processes. It is immediate, as (30) shows, that the more general form of the forward process can be readily constructed once the function u is specified and the market input processes A_t, Y_t and Z_t (which are independent of u) are chosen.

3.1 Dependence on the initial wealth

The explicit formulae (23) and (24) enable us to analyze the mappings $x \to X_t^*(\omega)$ and $x \to \pi_t^*(\omega)$, for fixed t and ω. We study this dependence next. To ease the presentation, we only discuss the case $Range\,(h) = (-\infty, +\infty)$. We also use the notation $X_t^{*,x}(\omega)$ and $\pi_t^{*,x}$, and introduce the function, $r : \mathbb{R} \times [0, +\infty) \to (0, +\infty)$, defined as

$$r(x, t) = h_x\left(h^{(-1)}(x, t), t\right). \tag{31}$$

A detailed discussion on its role, representation and differential properties is provided in section 4.5. Using (31), the optimal portfolio (cf. (24)) can be, then, written, for $t \geq 0$, as

$$\pi_t^{*,x} = r\left(X_t^{*,x}, A_t\right) \sigma_t^+ \lambda_t. \tag{32}$$

Proposition 5 *Let $X_t^{*,x}$ be given in (23), $t \geq 0$, and r as in (31). Then,*

$$\frac{\partial}{\partial x} X_t^{*,x} = \frac{r\left(X_t^{*,x}, A_t\right)}{r(x, 0)} \tag{33}$$

and

$$\frac{\partial}{\partial x} \pi_t^{*,x} = r_x\left(X_t^{*,x}, A_t\right) \frac{r\left(X_t^{*,x}, A_t\right)}{r(x, 0)} \sigma_t^+ \lambda_t. \tag{34}$$

Proof. Differentiating (23) with respect to x yields

$$\frac{\partial}{\partial x}X_t^{*,x} = h_x\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)\frac{\partial}{\partial x}h^{(-1)}(x,0),$$

and (33) follows from (31). To establish (34) we differentiate (32) and use (33). ■

The above result implies that the mapping $x \to X_t^{*,x}$ is increasing. This is to be expected because the larger the initial endowment the larger the future wealth should be. It, also, shows that the mapping $x \to \pi_t^{*,x}$ is increasing (or decreasing) depending on the monotonicity of the function r and the sign of λ_t. In general, the latter is not monotone and, therefore, nothing specific can be said about the dependence of the optimal allocation in terms of the initial endowment[3]. The monotonicity holds, however, in a special but frequently considered case, namely, when there is no bankruptcy, or more generally when the wealth stays always above a certain threshold. This case is considered in Proposition 23 herein where it is shown that $r_x \geq 0$ (see (65)). As a result, the mapping $x \to \pi_t^{*,x}$ is always increasing. Respectively, the other results in the same proposition show that the mapping $x \to \pi_t^{*,x}$ is always decreasing if the wealth stays below a threshold.

The optimal wealth formula (23) enables us to calculate higher order derivatives. For example, the second order derivative is given below.

Proposition 6 *Let $X_t^{*,x}$, $t \geq 0$, be given in (23) and r as in (31). Then,*

$$\frac{\partial^2}{\partial x^2}X_t^{*,x} = \frac{\left(r_x\left(X_t^{*,x}, A_t\right) - r_x(x,0)\right)}{r(x,0)}\frac{\partial}{\partial x}X_t^{*,x}.$$

Proof. Differentiating (33) yields

$$\frac{\partial^2}{\partial x^2}X_t^{*,x} = \frac{r_x\left(X_t^{*,x}, A_t\right)}{r(x,0)}\frac{\partial}{\partial x}X_t^{*,x} - \frac{r_x(x,0)}{r(x,0)}\frac{r\left(X_t^{*,x}, A_t\right)}{r(x,0)},$$

and we easily conclude using (33) once more. ■

Representation (23) reveals how the market input processes, A_t and M_t, $t \geq 0$, interact with the deterministic input, h, to generate the optimal wealth process. The function h is, on the other hand, fully specified by the measure ν. It is, then, natural to ask how the function h and, in turn, the process X_t^*, $t \geq 0$, depend on the total mass $\nu(\mathbb{R})$.

The result below shows an interesting scaling property which allows us to normalize the function h and assume that ν is a probability measure. For simplicity, we only discuss the case $Range(h) = (-\infty, +\infty)$.

Let $h_0 = \nu(\mathbb{R})$ and denote, with a slight abuse of notation, the associated wealth process by $X_t^*(x; h_0)$, $t \geq 0$.

[3] See, for example, the case $r(x,t) = \sqrt{ax^2 + be^{-at}}$ analyzed in [18].

Proposition 7 *For $h_0 = \nu(\mathbb{R})$, the optimal wealth process (cf. (23)) satisfies, for $t \geq 0$,*

$$\frac{1}{h_0} X_t^*(x; h_0) = X_t^*\left(\frac{x}{h_0}; 1\right).$$

Proof. Let $\overline{h}(x,t) = \frac{h(x,t)}{h_0}$. Then,

$$X_t^*(x; h_0) = h_0 \overline{h}\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right).$$

On the other hand, $h^{(-1)}(x,0) = \overline{h}^{(-1)}\left(\frac{x}{h_0}, 0\right)$ and, hence,

$$X_t^*(x; h_0) = h_0 \overline{h}\left(\overline{h}^{(-1)}\left(\frac{x}{h_0}, 0\right) + A_t + M_t, A_t\right) = h_0 X_t^*\left(\frac{x}{h_0}; 1\right).$$

■

4 Representation of the functions u and h

The functions u and h were instrumental in the construction of the forward performance, and the associated optimal wealth and portfolio processes (Theorem 4). In this section, we focus on the representation of these functions and connection with each other. We recall that they satisfy (14) and (19), respectively, and that we are interested in solutions of (14) that are increasing and strictly concave in their spatial argument. We will show that there is a *one-to-one* correspondence (modulo normalization constants) between these functions and strictly increasing solutions to the (backward) heat equation (19).

As it was discussed in the previous section (see representative results in Proposition 3)) the key idea is to represent h in terms of a finite positive Borel measure ν and, in turn, construct u from h. This measure, then, emerges as the defining element in the construction of any object of interest. The main assumption about it is that its bilateral Laplace transform exists[4]. Namely, we will be working throughout this section with measures belonging to $\mathcal{B}^+(\mathbb{R})$, given in (21). The connection between ν and h originates from the classical result of Widder (see, [17]) for nonnegative solutions of (19). For completeness and motivation we present this result below.

Theorem 8 *(Widder). Let $g(x,t)$, $(x,t) \in \mathbb{R} \times [0, +\infty)$, be a positive solution of (19). Then, there exists $\mu \in \mathcal{B}^+(\mathbb{R})$ such that g is represented as*

$$g(x,t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2 t} \mu(dy). \tag{35}$$

[4] We remind the reader that this condition is sufficient for the finiteness of h and u but does *not*, in general, guarantee admissibility of the associated policies. For the latter, condition (22) is used.

This result cannot be applied directly herein because, for the investment applications we consider, the wealth may not be assumed to remain always positive or, more generally, stay above (or below) a given threshold. As a consequence, different choices for the range of h, which represents the optimal wealth (cf. (23)), require different analysis. However, Widder's theorem will be applied to the function h_x which is positive (due to the assumed monotonicity of h) and also solves (19).

We start with the general theorem which gives us the representation of strictly increasing solutions to the heat equation (19). Its proof as well as all other proofs in this section are presented in an appendix.

We introduce the following sets,

$$\mathcal{B}_0^+(\mathbb{R}) = \left\{\nu \in \mathcal{B}^+(\mathbb{R}) \text{ and } \nu(\{0\}) = 0\right\}, \tag{36}$$

$$\mathcal{B}_+^+(\mathbb{R}) = \left\{\nu \in \mathcal{B}_0^+(\mathbb{R}) : \nu((-\infty, 0)) = 0 \right\} \tag{37}$$

and

$$\mathcal{B}_-^+(\mathbb{R}) = \left\{\nu \in \mathcal{B}_0^+(\mathbb{R}) : \ \nu((0, +\infty)) = 0 \right\}. \tag{38}$$

It is throughout assumed that the trivial case $\nu(\mathbb{R}) = 0$ is excluded.

In what follows C represents a generic constant. Special choices for it are discussed later on.

Proposition 9 *i) Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then, the function h defined, for $(x,t) \in \mathbb{R}\times[0,+\infty)$, by*

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - 1}{y} \nu(dy) + C, \tag{39}$$

is a strictly increasing solution to (19).

Moreover, if $\nu(\{0\}) > 0$, or $\nu \in \mathcal{B}_0^+(\mathbb{R})$, or $\nu \in \mathcal{B}_+^+(\mathbb{R})$ and $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, or $\nu \in \mathcal{B}_-^+(\mathbb{R})$ and $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} = -\infty$, then Range$(h) = (-\infty, +\infty)$, for $t \geq 0$.

On the other hand, if $\nu \in \mathcal{B}_+^+(\mathbb{R})$ with $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ (resp. $\nu \in \mathcal{B}_-^+(\mathbb{R})$ with $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty$)), then Range$(h) = \left(C - \int_{0^+}^{+\infty} \frac{\nu(dy)}{y}, +\infty\right)$ (resp. Range$(h) = \left(-\infty, C - \int_{-\infty}^{0^-} \frac{\nu(dy)}{y}\right)$), for $t \geq 0$.

ii) Conversely, let $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be a strictly increasing solution to (19). Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ such that h is given by (39).

Moreover, if Range$(h) = (-\infty, +\infty)$, $t \geq 0$, then it must be either that $\nu(\{0\}) > 0$, or $\nu \in \mathcal{B}_0^+(\mathbb{R})$, or $\nu \in \mathcal{B}_+^+(\mathbb{R})$ and $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, or $\nu \in \mathcal{B}_-^+(\mathbb{R})$ and $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} = -\infty$.

On the other hand, if Range$(h) = (x_0, +\infty)$ (resp. Range$(h) = (-\infty, x_0)$), $t \geq 0$ and $x_0 \in \mathbb{R}$, then it must be that $\nu \in \mathcal{B}_+^+(\mathbb{R})$ with $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ (resp. $\nu \in \mathcal{B}_-^+(\mathbb{R})$ with $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty$).

We continue with the representation of increasing and strictly concave solutions to (14). As mentioned earlier, we will show that there is a one to one correspondence (modulo normalization constants) between this class and the one of strictly increasing solutions to (19).

4.1 *Range* $(h) = (-\infty, +\infty)$

We recall that h is given by (39), for $(x,t) \in \mathbb{R} \times [0,+\infty)$. For convenience, we choose $C = 0$ and, thus,

$$h(0,0) = 0. \tag{40}$$

We show how to construct from such an h a globally defined, increasing and strictly concave solution u to (14). We, also, show the converse construction.

Note that from the properties of u, we would have $u_x(x,t) \neq 0$ and $|u(x,t)| < +\infty$, $t \geq 0$, for $(x,t) \in \mathbb{R} \times [0,+\infty)$. In addition, solutions of (14) are invariant with respect to affine transformations. Therefore, if u is a solution, the function

$$\hat{u}(x,t) = \frac{1}{u_x(x_0,0)} u(x,t) - \frac{u(x_0,0)}{u_x(x_0,0)}$$

is also a solution, for each $x_0 \in \mathbb{R}$. Without loss of generality, we may choose, as in (40), $x_0 = 0$ to be a reference point. We, then, assume that

$$u(0,0) = 0 \quad \text{and} \quad u_x(0,0) = 1. \tag{41}$$

Note, however, that while the first equality is imposed in an ad hoc way, the second one is in accordance with (40) (see (88) in proof of next proposition).

Proposition 10 *i) Let $\nu \in \mathcal{B}^+(\mathbb{R})$ and $h : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be as in (39) with the measure ν being used. Assume that h is of full range, for each $t \geq 0$, and let $h^{(-1)} : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined for $(x,t) \in \mathbb{R} \times [0,+\infty)$ and given by*

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{42}$$

is an increasing and strictly concave solution of (14) satisfying (41).

Moreover, for $t \geq 0$, the Inada conditions,

$$\lim_{x \to -\infty} u_x(x,t) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} u_x(x,t) = 0, \tag{43}$$

are satisfied.

ii) Conversely, let u be an increasing and strictly concave function satisfying, for $(x,t) \in \mathbb{R} \times [0,+\infty)$, (14) and (41), and the Inada conditions (43), for $t \geq 0$. Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R})$, such that u admits representation (42) with h given by (39), for $(x,t) \in \mathbb{R} \times [0,+\infty)$. Moreover, h is of full range, for each $t \geq 0$, and satisfies (40).

Example 11 *Let $\nu = \delta_0$, where δ_0 is a Dirac measure at* 0*. Then, (39) yields*

$$h(x,t) = x,$$

and, therefore, (42) implies

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-x+\frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x+\frac{t}{2}}.$$

This class of forward performance processes is analyzed in detail in [9].

Example 12 *Let $\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a})$, $a, b > 0$, and $\delta_{\pm a}$ are Dirac measures at $\pm a$. We, then, have*

$$h(x,t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax).$$

Thus,

$$h^{(-1)}(x,t) = \frac{1}{a} \ln\left(\frac{a}{b} x e^{\frac{1}{2}a^2 t} + \sqrt{\frac{a^2}{b^2} x^2 e^{a^2 t} + 1}\right)$$

and, in turn,

$$h_x\left(h^{(-1)}(x,t), t\right) = b e^{-\frac{1}{2}a^2 t} \cosh\left(\ln\left(\frac{a}{b} e^{\frac{1}{2}a^2 t} + \sqrt{\frac{a^2}{b^2} x^2 e^{a^2 t} + 1}\right)\right)$$

$$= \sqrt{a^2 x^2 + b^2 e^{-a^2 t}}.$$

If, $a = 1$, then (42) yields

$$u(x,t) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{e^t}{b^2} x \left(x - \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{t}{2}\right) - \frac{1}{2} \ln b,$$

while, if $a \neq 1$,

$$u(x,t)$$

$$= \frac{\sqrt[a]{a}}{a^2 - 1} e^{\frac{1-a}{2}t} \frac{b^2 e^{-at} + a(1+a)\left(ax^2 + x\sqrt{a^2 x^2 + b^2 e^{-a^2 t}}\right)}{\left(ax + \sqrt{a^2 x^2 + b^2 e^{-a^2 t}}\right)^{1+\frac{1}{\alpha}}} - \frac{\sqrt[a]{a}}{a^2 - 1} b^{1-\frac{1}{a}}.$$

The involved calculations are cumbersome and, for this, omitted. A complete description of this class of solutions can be found in [18].

It is worth mentioning that the above functions provide an interesting extension of the traditional power and logarithmic utilities, mostly frequently used in portfolio choice. Note, however, that the latter utilities are *not* globally defined while the above are.

Example 13 *Let $\nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$. Then,*

$$h(x,t) = F\left(\frac{x}{\sqrt{t+1}}\right) \quad \text{with} \quad F(x) = \int_0^x e^{\frac{1}{2}z^2} dz,\ x \in \mathbb{R}. \tag{44}$$

Therefore, $h^{(-1)}(x,t) = \sqrt{t+1} F^{(-1)}(x)$ and thus,

$$h_x\left(h^{(-1)}(x,t), t\right) = \frac{1}{\sqrt{t+1}} f\left(F^{(-1)}(x)\right), \tag{45}$$

with $f(x) = F'(x)$. Then, (42) becomes

$$u(x,t) = -\frac{1}{2}\int_0^t \frac{1}{\sqrt{s+1}} f\left(F^{(-1)}(x)\right) e^{-\sqrt{s+1}F^{(-1)}(x)+\frac{s}{2}} ds + \int_0^x e^{-F^{(-1)}(z)} dz.$$

It turns out that

$$u(x,t) = k_1 F\left(F^{(-1)}(x) - \sqrt{t+1}\right) + k_2 \tag{46}$$

with $k_1 = e^{-\frac{1}{2}}$ and $k_2 = e^{-\frac{1}{2}} \int_{-1}^0 e^{\frac{1}{2}z^2} dz$.

The calculations are rather tedious but one can verify that u satisfies (41) and solves (42). Indeed,

$$u_t(x,t) = -k_1 \frac{f\left(F^{(-1)}(x) - \sqrt{t+1}\right)}{2\sqrt{t+1}}, \qquad u_x(x,t) = k_1 \frac{f\left(F^{(-1)}(x) - \sqrt{t+1}\right)}{f\left(F^{(-1)}(x)\right)}$$

and

$$u_{xx}(x,t) = -k_1\sqrt{t+1}\frac{f\left(F^{(-1)}(x) - \sqrt{t+1}\right)}{\left(f\left(F^{(-1)}(x)\right)\right)^2},$$

and (14) follows. The equalities in (41) also follow from the form of u and the choice of the constants k_1, k_2. Note, moreover, that the above yields

$$u_x\left(F\left(\frac{x}{\sqrt{t+1}}\right), t\right) = e^{-x+\frac{t}{2}},$$

and (44) follows from (89).

From (24) and (45), we deduce that the optimal policy of the above example turns out to be

$$\pi_t^* = \frac{1}{\sqrt{A_t+1}} f\left(F^{(-1)}(X_t^*)\right) \sigma_t^+ \lambda_t, \tag{47}$$

with A_t, $t \geq 0$, as in (20) and

$$X_t^* = F\left(\frac{F^{(-1)}(x) + A_t + M_t}{\sqrt{A_t+1}}\right),$$

with the latter following from (23) and (44).

We can see that the above measure, $\nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$, violates condition (22) (for $t > 0$) and satisfies only (21). In turn, straightforward calculations show that π_t^*, $t \geq 0$, is admissible but only in the local sense, i.e., $\pi_t^* \in \mathcal{A}^l$ but $\pi_t^* \notin \mathcal{A}$. We, then, deduce that the process $U_t(x) = u(x, A_t)$ with u as in (46) satisfies Definition 2, of a local forward performance process.

4.2 *Range* $(h) = (x_0, +\infty)$, $x_0 \in \mathbb{R}$

We recall that in this case, h is given by (39) where ν satisfies

$$\nu \in \mathcal{B}_+^+(\mathbb{R}) \quad \text{and} \quad \int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty, \tag{48}$$

with $\mathcal{B}_+^+(\mathbb{R})$ given in (37). For convenience, we set in (39) $C = \int_{0^+}^{+\infty} \frac{1}{y}\nu(dy)$, yielding[5] *Range* $(h) = (0, +\infty)$, as

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2t}}{y}\nu(dy). \tag{49}$$

We can easily see, using the above and (14), that h is convex in its spatial argument and decreasing with regards to time,

$$h_{xx}(x,t) > 0 \quad \text{and} \quad h_t(x,t) < 0. \tag{50}$$

Next, we obtain the analogous to (42) representation of the differential input u. As (42) shows, h plays the role of the space argument of u. Thus, the latter is now defined on the half-line. Consideration then, needs, to be given to $\lim_{x\to 0} u(x,t)$, $t \geq 0$. The results below demonstrate that, depending on where the measure ν is concentrated, this limit can be finite or infinite. For the case of finite limit we have the following result.

Proposition 14 *i) Let ν satisfy (48) and, in addition, $\nu((0,1]) = 0$ and $\int_{1+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$[6]. Let, also, $h : \mathbb{R} \times [0,+\infty) \to (0,+\infty)$ be as in (49) and $h^{(-1)} : (0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined, for $(x,t) \in (0,+\infty) \times [0,+\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{51}$$

is an increasing and strictly concave solution of (14) with

$$\lim_{x\to 0} u(x,t) = 0, \quad \textit{for} \ \ t \geq 0. \tag{52}$$

Moreover, for $t \geq 0$, the Inada conditions

$$\lim_{x\to 0} u_x(x,t) = +\infty \quad \textit{and} \quad \lim_{x\to +\infty} u_x(x,t) = 0 \tag{53}$$

are satisfied.

ii) Conversely, let u, defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, be an increasing and strictly concave function satisfying (14), (52) and the Inada conditions (53). Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ satisfying (48), $\nu((0,1]) = 0$ and $\int_{1+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$, such that u admits representation (51) with h given by (49), for $(x,t) \in \mathbb{R} \times [0,+\infty)$.

[5] One may alternatively represent h as $h(x,t) = \int_0^{+\infty} e^{yx-\frac{1}{2}y^2t}\nu'(dy)$ with $\nu'(dy) = \frac{\nu(dy)}{y}$. Note that $\nu' \in \mathcal{B}^+(\mathbb{R})$. Such a representation was used in [1].

[6] The authors would like to thank an anonymous referee for pointing out that this integrability condition is needed.

Working along similar arguments we obtain the result covering the case of infinite limit.

Proposition 15 *i) Let ν satisfy (48) and, in addition, either $\nu((0,1]) > 0$ or $\nu((0,1]) = 0$ and $\int_{1+}^{+\infty} \frac{\nu(dy)}{y-1} = +\infty$. Let, also, $h : \mathbb{R} \times [0,+\infty) \to (0,+\infty)$ be as in (49) and $h^{(-1)} : (0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined, for $(x,t) \in (0,+\infty) \times [0,+\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_{x_0}^x e^{-h^{(-1)}(z,0)} dz, \quad (54)$$

for $x_0 > 0$, is an increasing and strictly concave solution of (14) with

$$\lim_{x\to 0} u(x,t) = -\infty, \quad \text{for } t \geq 0. \quad (55)$$

Moreover, for each $t \geq 0$, the Inada conditions

$$\lim_{x\to 0} u_x(x,t) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} u_x(x,t) = 0 \quad (56)$$

are satisfied.

ii) Conversely, let u, defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, be an increasing and strictly concave function satisfying (14), (55) and the Inada conditions (56). Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ satisfying (48) and $\nu((0,1]) > 0$ or (48) and $\nu((0,1]) = 0$ and $\int_{1+}^{+\infty} \frac{\nu(dy)}{y-1} = +\infty$, such that u admits representation (54) with h given, for $(x,t) \in \mathbb{R} \times [0,+\infty)$, by (49).

Example 16 *Let $\nu = \delta_\gamma$, $\gamma > 1$. Then, (49) yields*

$$h(x,t) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t},$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$. We, then, have $h^{(-1)}(x,t) = \ln(\gamma x)^{\frac{1}{\gamma}} + \frac{1}{2}\gamma t$, $(x,t) \in (0,+\infty) \times [0,+\infty)$ and, thus, $h_x\left(h^{(-1)}(x,t), t\right) = \gamma x$.

Since $\nu((0,1]) = 0$, u is given by (51) and, therefore,

$$u(x,t) = -\frac{1}{2}\int_0^t \gamma x e^{-\left(\ln(\gamma x)^{\frac{1}{\gamma}} + \frac{1}{2}\gamma s\right) + \frac{s}{2}} ds + \int_0^x (\gamma z)^{-\frac{1}{\gamma}} dz$$

$$= \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}t}.$$

Example 17 *Let $\nu = \delta_\gamma$, $\gamma = 1$. Then, (49) yields*

$$h(x,t) = e^{x - \frac{1}{2}t},$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$. We, then, have $h^{(-1)}(x,t) = \ln x + \frac{1}{2}t$, $(x,t) \in (0,+\infty) \times [0,+\infty)$ and, thus, $h_x\left(h^{(-1)}(x,t), t\right) = x$.

Since $\nu((0,1]) \neq 0$, u is given by (54). Therefore, for $(x,t) \in (0,+\infty) \times [0,+\infty)$ and $x_0 > 0$,

$$u(x,t) = -\frac{1}{2}\int_0^t x e^{-\left(\ln x + \frac{1}{2}s\right)+\frac{s}{2}} ds + \int_{x_0}^x \frac{1}{z} dz = \ln\frac{x}{x_0} - \frac{t}{2}.$$

Example 18 *Let $\nu = \delta_\gamma$, $\gamma \in (0,1)$. Using Example 16 and that $\nu((0,1]) \neq 0$, we easily deduce, using (54), that u is given by*

$$\begin{aligned} u(x,t) &= -\frac{1}{2}\int_0^t \gamma x e^{-\left(\ln(\gamma x)^{\frac{1}{\gamma}} + \frac{\gamma}{2}s\right)+\frac{s}{2}} ds + \int_{x_0}^x (\gamma z)^{-\frac{1}{\gamma}} dz \\ &= -\frac{\gamma^{\frac{\gamma-1}{\gamma}}}{1-\gamma} x^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{2}t} + \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{1-\gamma} x_0^{\frac{\gamma-1}{\gamma}}, \end{aligned}$$

for $x_0 > 0$.

4.3 *Range*$(h) = (-\infty, x_0)$, $x_0 \in \mathbb{R}$

We recall that in this case, h is given by (39) where ν satisfies

$$\nu \in \mathcal{B}_-^+(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty, \tag{57}$$

with $\mathcal{B}_-^+(\mathbb{R})$ given in (38). For convenience, we set in (39) $C = \int_{-\infty}^{0^-} \frac{\nu(dy)}{y}$, yielding *Range*$(h) = (-\infty, 0)$, as

$$h(x,t) = \int_{-\infty}^{0^-} \frac{e^{yx-\frac{1}{2}y^2 t}}{y} \nu(dy). \tag{58}$$

In analogy to (50), one can show that h is concave in its spatial argument and increasing with regards to time,

$$h_{xx}(x,t) < 0 \quad \text{and} \quad h_t(x,t) > 0. \tag{59}$$

The next proposition follows from a modification of the arguments used to prove Proposition 14.

Proposition 19 *i) Let ν be as in (57). Let, also, $h : \mathbb{R} \times [0,+\infty) \to (-\infty,0)$ be as in (58) and $h^{(-1)} : (-\infty,0) \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined, for $(x,t) \in (-\infty,0) \times [0,+\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds - \int_x^0 e^{-h^{(-1)}(z,0)} dz, \tag{60}$$

is an increasing and strictly concave solution of (14) with

$$\lim_{x\to 0} u(x,t) = 0, \quad \text{for } t \geq 0. \tag{61}$$

Moreover, for each $t \geq 0$, the Inada conditions

$$\lim_{x\to-\infty} u_x(x,t) = +\infty \qquad \text{and} \qquad \lim_{x\to0} u_x(x,t) = 0 \tag{62}$$

are satisfied.

ii) Conversely, let u be an increasing and strictly concave function satisfying, for $(x,t) \in (-\infty,0) \times [0,+\infty)$, (14), (61) and the Inada conditions (62), for $t \geq 0$. Then, there exists ν as in (57) such that u admits representation (60) with h given by (58).

Example 20 *We take $\nu = \delta_\gamma$, $\gamma = -\frac{1}{2k+1}$, $k > 0$. Then, (58) yields, for $(x,t) \in (-\infty,0) \times [0,+\infty)$,*

$$h(x,t) = -\frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t}.$$

Working as in Example 16, we deduce that, for $(x,t) \in (-\infty,0) \times [0,+\infty)$,

$$u(x,t) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}t} = -\frac{(2k+1)^{-2k-1}}{2(k+1)} x^{2(k+1)} e^{\frac{k+1}{2k+1}t}.$$

4.4 *Range*$(h) = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}$

The case of finite range is not considered since it does not yield a meaningful solution. Indeed, we recall the following result derived by Widder (see [17]).

Proposition 21 *Let h be a solution to (19) such that for $(x,t) \in \mathbb{R}\times[0,\infty)$, $-M \leq h(x,t) \leq M$, for some constant M. Then, $h(x,t)$ is constant.*

It, then, easily follows that in this case the problem degenerates as there is *no* strictly increasing solution to (19) and, in turn to (14).

4.5 The local risk tolerance

In the previous section we introduced the function r in (31). This function facilitates the representation of the optimal portfolio policy (cf. (32)) and, as it is shown below, is represented in terms of the spatial derivatives of u. In the traditional maximal expected utility models, a similar quantity is used, known as the risk tolerance. We keep an analogous terminology herein.

For the generic spatial domain $\mathbb{D}$ appearing below, we have $\mathbb{D} = \mathbb{R}$, $(0,+\infty)$ or $(-\infty,0)$. To ease the presentation, we omit any reference to the specific range h (and, thus, to the domain of u).

The following result is a direct consequence of (31) and (89) (for an alternative proof, see [10]).

Proposition 22 *Let $r : \mathbb{D}\times[0,\infty) \to (0,+\infty)$ be given by (31), i.e.*

$$r(x,t) = h_x\left(h^{(-1)}(x,t), t\right), \tag{63}$$

and u be the associated with h differential utility input. Then, for $(x,t) \in \mathbb{D}\times[0,\infty)$,

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}.$$

Therefore, $r(x,t) = R(x,t)$ with $R(x,t)$ as in (15).

In (34) we saw that the monotonicity of the optimal investment strategy with regards to the initial endowment depends directly on the sign of the partial derivative $r_x(x,t)$. As it was mentioned in section 3, when the risk tolerance is defined on $\mathbb{R}\times[0,+\infty)$ very little, if anything, can be established for its monotonicity or limiting behavior. When, however, its domain is semi-infinite, we have the following results.

Proposition 23 *Let $r : \mathbb{D}\times[0,+\infty) \to (0,+\infty)$ with $\mathbb{D} = (0,+\infty)$ or $(-\infty,0)$. Then, for $t \geq 0$,*

$$\lim_{x\to 0} r(x,t) = 0. \tag{64}$$

If $\mathbb{D} = (0,+\infty)$, then

$$r_x(x,t) \geq 0, \tag{65}$$

while, if $\mathbb{D} = (-\infty,0)$,

$$r_x(x,t) \leq 0, \tag{66}$$

for $t \geq 0$.

Proof. We only establish (64) when $\mathbb{D} = (0,+\infty)$. Recalling that

$$h_x(x,t) = \int_{0^+}^{+\infty} e^{yx-\frac{1}{2}y^2 t}\nu(dy)$$

(cf. (49)), (63) yields

$$r(x,t) = \int_{0^+}^{\infty} e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\nu(dy).$$

Passing to the limit, and using the monotone convergence theorem and (93), we conclude.

Next, we show (65). Differentiating (63) yields

$$r_x(x,t) = \left(\frac{\partial}{\partial x}h^{(-1)}(x,t)\right) h_{xx}\left(h^{(-1)}(x,t), t\right).$$

When $\mathbb{D} = (0,+\infty)$ (resp. $\mathbb{D} = (-\infty,0)$), then (65) (resp. (66)) follows from (50) (resp. (59)). ■

5 Deterministic market prices of risk

In this section we assume that the process $\lambda_t, t \geq 0$, (cf. (3)) is deterministic. This, in turn, yields that A_t, $t \geq 0$, (see (20)) is deterministic.

The goal is twofold. Firstly, we study distributional properties of the optimal wealth and compute its cumulative distribution, density and moments. Secondly, we explore some inverse problems, namely, how could the investor's preferences be inferred from information about the targeted mean of his optimal wealth.

5.1 Distribution of the optimal wealth process

Recalling A_t and M_t from (20) we have $\langle M \rangle_t = A_t$, $t \geq 0$, and, thus, by Levy's theorem the process M_t is a Gaussian martingale. This leads to the following properties of the distribution of the investor's optimal wealth process. The functions N and n below stand, respectively, for the cumulative distribution and the density functions of a standard normal variable. We recall that h solves (19), $h^{(-1)}$ stands for its spatial inverse and r is given in (31).

Proposition 24 *i) The cumulative distribution and probability density functions of the optimal wealth $X_t^{*,x}$, $t > 0$, are given, respectively, by*

$$\mathbb{P}\left(X_t^{*,x} \leq y\right) = N\left(\frac{h^{(-1)}\left(y, A_t\right) - h^{(-1)}\left(x, 0\right) - A_t}{\sqrt{A_t}}\right) \tag{67}$$

and

$$f_{X_t^{*,x}}\left(y\right) = n\left(\frac{h^{(-1)}\left(y, A_t\right) - h^{(-1)}\left(x, 0\right) - A_t}{\sqrt{A_t}}\right) \frac{1}{r\left(y, A_t\right)\sqrt{A_t}},$$

with A_t as in (20).

ii) For all $p \in [0,1]$ and $t > 0$, the quantile of order p, i.e. the point $y_p(t)$ for which $\mathbb{P}\left(X_t^{*,x} \leq y_p(t)\right) = p$, is given by

$$y_p\left(t\right) = h\left(h^{(-1)}\left(x, 0\right) + A_t + \sqrt{A_t} N^{(-1)}\left(p\right), A_t\right).$$

Proof. The first statement follows directly from (23). Indeed,

$$\begin{aligned}\mathbb{P}\left(X_t^{*,x} \leq y\right) &= \mathbb{P}\left(h\left(h^{(-1)}\left(x, 0\right) + A_t + M_t, A_t\right) \leq y\right) \\ &= \mathbb{P}\left(h^{(-1)}\left(x, 0\right) + A_t + M_t \leq h^{(-1)}\left(y, A_t\right)\right),\end{aligned}$$

and we easily conclude. The other two statements are also immediate. ■

Properties of the multivariate distributions may be analyzed along similar arguments.

Next, we study the expected value of the optimal wealth and portfolio processes.

Proposition 25 *Let $X_t^{*,x}$ and $\pi_t^{*,x}$ be as in (23) and (24). Then, for $t > 0$,*

$$E\left(X_t^{*,x}\right) = h\left(h^{(-1)}(x,0) + A_t, 0\right). \tag{68}$$

Moreover,

$$\frac{\partial}{\partial x} E\left(X_t^{*,x}\right) = \frac{r\left(E\left(X_t^{*,x}\right), 0\right)}{r(x,0)} \tag{69}$$

and

$$E\left(r\left(X_t^{*,x}, A_t\right)\right) = r\left(E\left(X_t^{*,x}\right), 0\right). \tag{70}$$

Proof. We establish the above when the function h is as in (39), with the other cases, exhibited in (49) and (58), following along similar arguments.

From (23) we have

$$E\left(X_t^{*,x}\right) = E\left(h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)\right)$$

$$= E\left(\int_{\mathbb{R}} \frac{e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2 A_t} - 1}{y} \nu(dy)\right)$$

$$= \int_{\mathbb{R}} E\left(\frac{e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2 A_t} - 1}{y}\right) \nu(dy)$$

$$= \int_{\mathbb{R}} \frac{e^{y\left(h^{(-1)}(x,0)+A_t\right)} - 1}{y} \nu(dy) = h\left(h^{(-1)}(x,0) + A_t, 0\right).$$

Above we used that the two integrals can be interchanged. For this, it suffices to have

$$E\left(\int_{\mathbb{R}} \left| \frac{e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2 A_t} - 1}{y} \right| \nu(dy)\right) < +\infty. \tag{71}$$

Indeed, inequality (79) yields

$$\int_{\mathbb{R}} \left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \nu(dy) \le \nu(\mathbb{R}) \left(e^{x+\frac{t}{2}} + e^{-x+\frac{t}{2}}\right) + \int_{\mathbb{R}} e^{yx-\frac{1}{2}y^2 t} \nu(dy).$$

Therefore,

$$E\left(\int_{\mathbb{R}} \left| \frac{e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2 A_t} - 1}{y} \right| \nu(dy)\right)$$

$$\le \nu(\mathbb{R})\, E\left(e^{h^{(-1)}(x,0)+A_t+M_t+\frac{A_t}{2}} + e^{-\left(h^{(-1)}(x,0)+A_t+M_t\right)+\frac{A_t}{2}}\right)$$

$$+E\left(\int_{\mathbb{R}} e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2 A_t} \nu(dy)\right).$$

On the other hand,

$$E\left(e^{h^{(-1)}(x,0)+A_t+M_t+\frac{A_t}{2}}+e^{-\left(h^{(-1)}(x,0)+A_t+M_t\right)+\frac{A_t}{2}}\right)$$

$$\leq e^{h^{(-1)}(x,0)+\frac{3}{2}c^2 t}E\left(e^{M_t}\right)+e^{-h^{(-1)}(x,0)},$$

where we used (5) and (20). Similarly,

$$E\left(\int_{\mathbb{R}}e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2A_t}\nu\left(dy\right)\right)$$

$$=\int_{\mathbb{R}}E\left(e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2A_t}\right)\nu\left(dy\right)$$

$$=\int_{\mathbb{R}}e^{y\left(h^{(-1)}(x,0)+A_t\right)}\nu\left(dy\right)<\int_{\mathbb{R}}e^{y\left(h^{(-1)}(x,0)+c^2t\right)}\nu\left(dy\right)<+\infty,$$

and we easily obtain (71). Assertion (69) follows from (68) and (31).

To show (70), we recall (23), (31) and (81) yielding

$$E\left(r\left(X_t^{*,x},A_t\right)\right)=E\left(\int_{\mathbb{R}}e^{y\left(h^{(-1)}(x,0)+A_t+M_t\right)-\frac{1}{2}y^2A_t}\nu\left(dy\right)\right)$$

$$=\int_{\mathbb{R}}e^{y\left(h^{(-1)}(x,0)+A_t\right)}\nu\left(dy\right)=h_x\left(h^{(-1)}\left(x,0\right)+A_t,0\right),$$

and we easily conclude. ■

5.2 Inferring the investor's preferences

The investment performance criterion (27) combines the investor's preferences with the market input. As a consequence, the optimal portfolio and the associated wealth (see (24) and (23), respectively) contain implicit information about these preferences. In this section, we discuss how to learn about the individual's risk attitude by analyzing distributional characteristics of his optimal wealth. One can say, using the language of the derivatives industry, that our aim is to *calibrate* the investor's preferences, given the market dynamics and his desirable distributional outcomes for his wealth process.

This idea is relatively new. To the best of our knowledge, the authors of [15] were the first to propose a model and show how information about an investor's marginal utility of wealth can be inferred from her choice of a distribution. Other, more recent relevant references, are [2] and [16].

We discuss two examples in which we infer the investor's preferences using information about the behavior of her average future wealth. For simplicity, we only concentrate on the no-bankruptcy case, $Range(h)=(0,+\infty)$ (see section 4.2).

We remind the reader that the market price of risk is taken to be deterministic. As a result, A_t, $t\geq 0$, (cf. (20)) is also deterministic.

Proposition 26 *Let the mapping $x \to E\left(X_t^{*,x}\right)$ be linear, for all $x > 0$ and $t \geq 0$. Then, there exists a positive constant $\gamma > 0$ such that the investor's forward performance process is given by*

$$U_t(x) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{1}{2}(\gamma-1)A_t}, \tag{72}$$

if $\gamma \neq 1$ and by

$$U_t(x) = \ln x - \frac{1}{2} A_t, \tag{73}$$

if $\gamma = 1$. Moreover,

$$E\left(X_t^{*,x}\right) = x e^{\gamma A_t}. \tag{74}$$

Proof. Differentiating (69) we deduce

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} E\left(X_t^{*,x}\right) &= \frac{r_x\left(E\left(X_t^{*,x}\right),0\right)}{r(x,0)} \frac{\partial}{\partial x} E\left(X_t^{*,x}\right) - r\left(E\left(X_t^{*,x}\right),0\right) \frac{r_x(x,0)}{r^2(x,0)} \\
&= \frac{\left(r_x\left(E\left(X_t^{*,x}\right),0\right) - r_x(x,0)\right)}{r^2(x,0)} r\left(E\left(X_t^{*,x}\right),0\right).
\end{aligned}$$

By assumption $\frac{\partial^2}{\partial x^2} E\left(X_t^{*,x}\right) = 0$. Moreover, $r\left(E\left(X_t^{*,x}\right),0\right) > 0$ as it follows from (70). Therefore, we must have

$$r_x\left(E\left(X_t^{*,x}\right),0\right) = r_x(x,0)$$

and, in turn,

$$\frac{\partial}{\partial t} r_x\left(E\left(X_t^{*,x}\right),0\right) = r_{xx}\left(E\left(X_t^{*,x}\right),0\right) \frac{\partial}{\partial t} E\left(X_t^{*,x}\right) = 0.$$

However, (68) implies $\frac{\partial}{\partial t} E\left(X_t^{*,x}\right) \neq 0$ and, thus, we deduce that

$$r_{xx}\left(E\left(X_t^{*,x}\right),0\right) = 0.$$

Therefore, the function $r(x,0)$ must be linear in $E\left(X_t^{*,x}\right)$ and, in turn, in x, per our assumption. Using (64) we obtain

$$r(x,0) = \gamma x,$$

for some $\gamma > 0$. From (31) and (49) we, then, deduce that for all $x > 0$,

$$\int_{0^+}^{+\infty} e^{y h^{(-1)}(x,0)} \nu(dy) = \gamma x$$

and, in turn,

$$\int_{0^+}^{+\infty} e^{yx} \nu(dy) = \gamma \int_{0^+}^{+\infty} \frac{e^{yx}}{y} \nu(dy).$$

Therefore, we must have

$$\nu(dy) = \delta_\gamma$$

where δ_γ is a Dirac measure at $\gamma > 0$. This yields

$$h(x,t) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t}$$

and assertions (72) and (73) follow from Examples 16,17,18 and 20.

Equality (74) follows from (68) and the form of h. ■

From the above analysis, we see that calibrating the investor's preferences consists of choosing a time horizon and the level of the mean of her optimal wealth, say t_0 and mx ($m > 1$), respectively. Then, (74) implies that the corresponding γ must satisfy $xe^{\gamma A_{t_0}} = mx$, or, equivalently,

$$\gamma = \frac{\ln m}{A_{t_0}}.$$

Note that under the linearity assumption, the investor can calibrate her expectations only for *a single* time horizon. The model interpolates for all other trading horizons, giving

$$E\left(X_t^{*x}\right) = xm^{\frac{A_t}{A_{t_0}}}.$$

We easily deduce that the distribution of the optimal wealth $X_t^{*,x}$ is lognormal, for all (x,t).

The linearity of the mapping $x \to E\left(X_t^{*,x}\right)$ is a very strong assumption. Indeed, it only allows for calibration of a single parameter, namely, the slope, and, moreover, for a single time horizon. Therefore, if one intends to calibrate the investor's preferences to more refined information, then one needs to accept a more complicated dependence of $E\left(X_t^{*,x}\right)$ on x. We discuss this case next.

To this end, let us fix the level of initial wealth at $x = 1$ and consider calibration to $E\left(X_t^{*,1}\right)$, for $t > 0$. The investor then chooses an increasing function $m(t)$ (with $m(t) > 1$) to represent the latter, i.e. for $t > 0$,

$$E\left(X_t^{*,1}\right) = m(t). \tag{75}$$

What does this choice reveal about her preferences? Moreover, can she choose an arbitrary increasing function $m(t)$? We give answers to these questions below.

In analogy to the previous proposition, we only consider the no bankruptcy case, which corresponds to h given in (49). Using arguments similar to the ones used in Proposition 7, we may assume, without loss of generality, that $\int_0^{+\infty} \frac{\nu(dy)}{y} = 1$. We, then, have $h^{(-1)}(1,0) = 0$ which, combined with (68), yields

$$E\left(X_t^{*,1}\right) = h(A_t, 0) = \int_{0^+}^{\infty} \frac{e^{yA_t}}{y} \nu(dy).$$

We easily see that the investor may only choose a function $m(t)$, $t \geq 0$, which can be represented in the form

$$m(t) = \int_{0^+}^{\infty} \frac{e^{yA_t}}{y} \nu(dy). \tag{76}$$

Therefore, the choice of a feasible $m(t)$ reduces to the choice of the appropriate measure ν. Specifically, we must have,

$$m\left(A_t^{(-1)}\right) = \int_{0^+}^{\infty} \frac{e^{yt}}{y} \nu(dy) = h(t, 0), \tag{77}$$

where $A_t^{(-1)}$ stands for the inverse of the function A_t, $t \geq 0$. Note that the right hand side above is the moment generating function of the probability measure $\mu(dy) = \frac{\nu(dy)}{y}$.

Assume now that the mean $m(t)$ (cf. (75)) was chosen so that (77) holds for some measure ν. Then, for other values of $x > 0$, $x \neq 1$, we have, using (68),

$$\begin{aligned} E\left(X_t^{*,x}\right) &= m\left(A^{(-1)}\left(h^{(-1)}(x,0) + A(t)\right)\right) \\ &= m\left(A^{(-1)}\left(A\left(m^{(-1)}(x)\right) + A(t)\right)\right), \end{aligned}$$

where, for notational convenience, A_t (resp. $A_t^{(-1)}$) is denoted by $A(t)$ (resp. $A^{(-1)}(t)$).

In summary, the market related input A_t, coupled with the investor's targeted mean $m(t)$ (at initial wealth $x = 1$), yields the investor's preferences by choosing the measure ν, appearing in (76). Note that the function A_t determines the distribution of the martingale M_t and the measure ν defines the function h, which, in turn, determines the functions u and r. Once these quantities are specified, we are able to construct the optimal portfolio process that generates the optimal wealth which satisfies (75). The distribution of the optimal wealth process $X_t^{*,x}$, for $x > 0$, $x \neq 1$, is, in turn, deduced from the specification of the targeted mean, $m(t)$, the market input A_t and (67).

We conclude mentioning that one may prefer to *calibrate the distribution* of the optimal wealth at a given time, say, $X_{t_0}^{*,1}$, rather than the mean, $E\left(X_t^{*,1}\right)$, for $t \geq 0$. It is easy to see what distributions are attainable. Indeed, Proposition 24 would give

$$\mathbb{P}\left(X_{t_0}^{*,1} \leq y\right) = N\left(\frac{h^{(-1)}(y, A_{t_0}) - A_{t_0}}{\sqrt{A_{t_0}}}\right)$$

with

$$h(y, A_{t_0}) = \int_{0^+}^{\infty} \frac{e^{zy - \frac{1}{2}z^2 A_{t_0}}}{y} \nu(dz).$$

Further analysis of this and other calibration issues is left for future research.

6 Appendix

Proof of Proposition 9: i) Without loss of generality, we take $C = 0$. We first establish that $h(x,t)$ is well defined. Indeed, for $(x,t) \in \mathbb{R} \times [0,+\infty)$, we have

$$\int_{\mathbb{R}} \left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \nu(dy) = \int_{|y|>1} \left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \nu(dy) + \int_{|y|\leq 1} \left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \nu(dy). \tag{78}$$

On the other hand, one can show that, for fixed (x,t) and $|y| \leq 1$, the inequality

$$\left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \leq e^{|x|+\frac{t}{2}} - 1 \tag{79}$$

holds[7]. Combining the above, we deduce

$$\int_{\mathbb{R}} \left| \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \right| \nu(dy) \leq \int_{|y|>1} e^{yx} \nu(dy) + \nu(\mathbb{R})\, e^{|x|+\frac{t}{2}} < +\infty. \tag{80}$$

Differentiating under the integral yields

$$h_x(x,t) = \int_{\mathbb{R}} e^{yx-\frac{1}{2}y^2 t} \nu(dy) \tag{81}$$

and the claimed monotonicity of h follows. Note that $h_x(x,t)$ is well defined because $0 \leq h_x(x,t) < h_x(x,0) < +\infty$, as $\nu \in \mathcal{B}^+(\mathbb{R})$. Further differentiation yields

$$h_{xx}(x,t) = \int_{\mathbb{R}} y e^{yx-\frac{1}{2}y^2 t} \nu(dy) \quad \text{and} \quad h_t(x,t) = -\frac{1}{2} \int_{\mathbb{R}} y e^{yx-\frac{1}{2}y^2 t} \nu(dy).$$

The fact that h solves (19) would follow provided the above integrals are well defined. For $x \neq 0$, we have

$$\left| \int_{\mathbb{R}} y e^{yx-\frac{1}{2}y^2 t} \nu(dy) \right| \leq \frac{1}{|x|} \int_{\mathbb{R}} |yx|\, e^{yx-\frac{1}{2}y^2 t} \nu(dy)$$

$$\leq \frac{1}{|x|} \int_{\mathbb{R}} \left(e^{|yx|} - 1 \right) e^{yx-\frac{1}{2}y^2 t} \nu(dy)$$

$$\leq \frac{1}{|x|} \left(\int_{yx\leq 0} \left(e^{|yx|} - 1 \right) e^{yx-\frac{1}{2}y^2 t} \nu(dy) + \int_{yx>0} \left(e^{|yx|} - 1 \right) e^{yx-\frac{1}{2}y^2 t} \nu(dy) \right)$$

---

[7] Indeed, we have $\frac{e^{yx-\frac{1}{2}y^2 t}-1}{y} = \sum_{n=1}^{\infty} \frac{y^{n-1}(x-\frac{1}{2}yt)^n}{n!}$ and, therefore, $\left| \frac{e^{yx-\frac{1}{2}y^2 t}-1}{y} \right| \leq \sum_{n=1}^{\infty} \frac{|y|^{n-1}|x-\frac{1}{2}yt|^n}{n!} \leq \sum_{n=1}^{\infty} \frac{(|x|+\frac{1}{2}|y|t)^n}{n!} \leq e^{|x|+\frac{t}{2}} - 1$.

$$\leq \frac{1}{|x|}\left(\int_{yx\leq 0}\left(1-e^{yx}\right)e^{-\frac{1}{2}y^2t}\nu\left(dy\right)+\int_{yx>0}e^{2yx-\frac{1}{2}y^2t}\nu\left(dy\right)\right)$$
$$\leq \frac{1}{|x|}\int_{yx\leq 0}\nu\left(dy\right)+\frac{1}{|x|}\int_{yx>0}e^{2yx}\nu\left(dy\right),$$
and the assertion follows using that $\nu\in\mathcal{B}^{+}\left(\mathbb{R}\right)$. The case $x=0$ follows trivially.

Next, we establish that if ν has the aforestated properties then, for each $t\geq 0$, h is of full range. Given that h is continuous, we need to show that, for $t\geq 0$,

$$\lim_{x\to-\infty}h\left(x,t\right)=-\infty \quad\text{ and }\quad \lim_{x\to+\infty}h\left(x,t\right)=+\infty. \tag{82}$$

From (39) we have

$$h\left(x,t\right)=\int_{-\infty}^{0^-}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right)+x\nu\left(\{0\}\right)+\int_{0^+}^{+\infty}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right). \tag{83}$$

We first look at the case $\nu\left(\{0\}\right)>0$. If both $\nu\left(\left(-\infty,0\right)\right)=0$ and $\nu\left(\left(0,+\infty\right)\right)=0$, (82) follows directly. If $\nu\left(\left(-\infty,0\right)\right)=0$ and $\nu\left(\left(0,+\infty\right)\right)>0$, the monotone convergence theorem yields

$$\lim_{x\to\pm\infty}\left(x\nu\left(\{0\}\right)+\int_{0^+}^{+\infty}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right)\right)=\pm\infty.$$

The case $\nu\left(\left(-\infty,0\right)\right)>0$ and $\nu\left(\left(0,+\infty\right)\right)=0$ follows similarly.

Next, we assume $\nu\in\mathcal{B}_0^{+}\left(\mathbb{R}\right)$. Then, (83) yields

$$h\left(x,t\right)=\int_{-\infty}^{0^-}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right)+\int_{0^+}^{+\infty}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right),$$

and we easily deduce (82) if $\nu\left(\left(-\infty,0\right)\right)\times\nu\left(\left(0,+\infty\right)\right)>0$.

If $\nu\in\mathcal{B}_0^{+}\left(\mathbb{R}\right)$ and it, also, satisfies $\nu\left(\left(-\infty,0\right)\right)=0$ and $\int_{0^+}^{+\infty}\frac{1}{y}\nu\left(dy\right)=+\infty$, then the monotone convergence theorem yields

$$\lim_{x\to+\infty}h\left(x,t\right)=\lim_{x\to+\infty}\int_{0^+}^{+\infty}\frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu\left(dy\right)=+\infty$$

and

$$\lim_{x\to-\infty}h\left(x,t\right)=-\int_{0^+}^{+\infty}\frac{1}{y}\nu\left(dy\right)=-\infty.$$

The case $\nu\in\mathcal{B}_{-}^{+}\left(\mathbb{R}\right)$ with $\int_{-\infty}^{0^-}\frac{1}{y}\nu\left(dy\right)=-\infty$ follows similarly as well as the cases $\nu\in\mathcal{B}_{+}^{+}\left(\mathbb{R}\right)$ with $\int_{0^+}^{+\infty}\frac{1}{y}\nu\left(dy\right)<+\infty$, and $\nu\in\mathcal{B}_{-}^{+}\left(\mathbb{R}\right)$ with $\int_{-\infty}^{0^-}\frac{1}{y}\nu\left(dy\right)>-\infty$.

ii) Let h be a strictly increasing solution to (19). Then, its spatial derivative satisfies $h_x\left(x,t\right)\geq 0$ and solves (19). Thus, Widder's theorem implies the existence of $\nu\in\mathcal{B}^{+}\left(\mathbb{R}\right)$ such that the representation

$$h_x\left(x,t\right)=\int_{\mathbb{R}}e^{yx-\frac{1}{2}y^2t}\nu\left(dy\right) \tag{84}$$

holds. We, then, have $h_{xx}(x,t) = \int_{\mathbb{R}} ye^{yx-\frac{1}{2}y^2t}\nu(dy)$ (its finiteness follows easily) which combined with (19) yields

$$h_t(x,t) = -\frac{1}{2}\int_{\mathbb{R}} ye^{yx-\frac{1}{2}y^2t}\nu(dy).$$

If $Range(h) = (-\infty, +\infty)$, $t \geq 0$, integrating yields

$$h(x,t) = \int_0^t h_t(x,s)\,ds + \int_{x_0}^x h_x(z,0)\,dz + h(x_0,0), \tag{85}$$

for any $x_0 \in \mathbb{R}$. Combining the above we obtain

$$h(x,t) = -\frac{1}{2}\int_0^t \int_{\mathbb{R}} ye^{yx-\frac{1}{2}y^2s}\nu(dy)\,ds + \int_{x_0}^x \int_{\mathbb{R}} e^{yz}\nu(dy)\,dz + h(x_0,0). \tag{86}$$

Note that for $\nu \in \mathcal{B}^+(\mathbb{R})$,

$$\int_0^t \int_{\mathbb{R}} \left| ye^{yx-\frac{1}{2}y^2s} \right| \nu(dy)\,ds \leq t\int_{\mathbb{R}} |y|\,e^{yx}\nu(dy) < \infty,$$

and, thus, Fubini's theorem yields

$$\begin{aligned}
-\frac{1}{2}\int_0^t \int_{\mathbb{R}} ye^{yx-\frac{1}{2}y^2s}\nu(dy)\,ds &= -\frac{1}{2}\int_{\mathbb{R}}\int_0^t ye^{yx-\frac{1}{2}y^2s}ds\nu(dy) \\
&= \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - e^{yx}}{y}\nu(dy).
\end{aligned}$$

Moreover, Tonelli's theorem yields

$$\int_{x_0}^x \int_{\mathbb{R}} e^{yz}\nu(dy)\,dz = \int_{\mathbb{R}}\int_{x_0}^x e^{yz}dz\nu(dy) = \int_{\mathbb{R}} \frac{e^{yx} - e^{yx_0}}{y}\nu(dy).$$

Observe that both integrals above are well defined as it was shown in the proof of part i). Using (86) gives

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - e^{yx_0}}{y}\nu(dy) + h(x_0,0).$$

Without loss of generality we choose $x_0 = 0$ and we easily conclude.

Next, we establish that if h is of full range, for each $t \geq 0$, it must be that $\nu(\{0\}) > 0$, or, otherwise, either $\nu \in \mathcal{B}_0^+(\mathbb{R})$, or $\nu \in \mathcal{B}_+^+(\mathbb{R})$ with $\int_{0^+}^{+\infty} \frac{1}{y}\nu(dy) = +\infty$, or $\nu \in \mathcal{B}_-^+(\mathbb{R})$ with $\int_{-\infty}^{0^-} \frac{1}{y}\nu(dy) = -\infty$. Note that (82) must hold, for each $t \geq 0$, as h is continuous.

Let us assume that $\nu \in \mathcal{B}_+^+(\mathbb{R})$ and $\int_{0^+}^{+\infty} \frac{1}{y}\nu(dy) < +\infty$. Then, (83) would give

$$\lim_{x\to-\infty} h(x,t) = \lim_{x\to-\infty} \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2t} - 1}{y}\nu(dy) = -\int_{0^+}^{+\infty} \frac{1}{y}\nu(dy) > -\infty,$$

contradicting (82). All other cases follow along similar arguments and their proof is, thus, omitted.

The following auxiliary result will be used in the sequel. Because we will examine the various cases of the range of h separately, we state the result without making specific reference to the domain of the spatial inverse, $h^{(-1)}$.

Lemma 27 *A strictly increasing function, say h, satisfies (19) if and only if its spatial inverse, $h^{(-1)}$, satisfies*

$$\frac{\partial}{\partial t}h^{(-1)}(x,t)+\frac{1}{2}\frac{\frac{\partial^2}{\partial x^2}h^{(-1)}(x,t)}{\left(\frac{\partial}{\partial x}h^{(-1)}(x,t)\right)^2}=0. \tag{87}$$

We continue with the proofs of Propositions 10, 14 and 15.

Proof of Proposition 10: i) First, we establish that the integrals in (42) are well defined. Using (84) and Tonelli's theorem we have

$$\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}}h_x\left(h^{(-1)}(x,s),s\right)ds=\int_0^t\int_{\mathbb{R}}e^{(y-1)h^{(-1)}(x,s)+\frac{s}{2}-\frac{1}{2}y^2s}\nu(dy)\,ds$$

$$\leq e^{\frac{t}{2}}\int_{\mathbb{R}}\int_0^t e^{(y-1)h^{(-1)}(x,s)}ds\nu(dy)$$

$$=e^{\frac{t}{2}}\int_{y\geq 1}\int_0^t e^{(y-1)h^{(-1)}(x,s)}ds\nu(dy)+e^{\frac{t}{2}}\int_{y<1}\int_0^t e^{(y-1)h^{(-1)}(x,s)}ds\nu(dy)$$

$$\leq te^{\frac{t}{2}}\int_{y\geq 1}e^{(y-1)\max_{0\leq s\leq t}h^{(-1)}(x,s)}\nu(dy)$$

$$+te^{\frac{t}{2}}\int_{y<1}e^{(y-1)\min_{0\leq s\leq t}h^{(-1)}(x,s)}\nu(dy).$$

Using Tonelli's theorem once more, we have that the second integral in (42) satisfies

$$\int_0^x e^{-h^{(-1)}(z,0)}dz=\int_{h^{(-1)}(0,0)}^{h^{(-1)}(x,0)}e^{-z'}h_x(z',0)\,dz'$$

$$=\int_{h^{(-1)}(0,0)}^{h^{(-1)}(x,0)}\int_{\mathbb{R}}e^{(y-1)z'}\nu(dy)\,dz'=\int_{\mathbb{R}}\frac{e^{(y-1)h^{(-1)}(x,0)}-e^{(y-1)h^{(-1)}(0,0)}}{y-1}\nu(dy),$$

and its finiteness follows from arguments similar to the ones used in the proof of Proposition 9.

Differentiating (42) and using that h solves (19) yield

$$u_x(x,t)=\frac{1}{2}\int_0^t\left(1-\frac{h_{xx}\left(h^{(-1)}(x,s),s\right)}{h_x\left(h^{(-1)}(x,s),s\right)}\right)e^{-h^{(-1)}(x,s)+\frac{s}{2}}ds+e^{-h^{(-1)}(x,0)}$$

$$= \frac{1}{2} \int_0^t \left(1 + 2 \frac{h_t \left(h^{(-1)} (x,s), s \right)}{h_x \left(h^{(-1)} (x,s), s \right)} \right) e^{-h^{(-1)}(x,s) + \frac{s}{2}} ds + e^{-h^{(-1)}(x,0)}$$

$$= \int_0^t \left(\frac{1}{2} - \frac{\partial}{\partial s} h^{(-1)} (x,s) \right) e^{-h^{(-1)}(x,s) + \frac{s}{2}} ds + e^{-h^{(-1)}(x,0)}$$

and, therefore,

$$u_x (x,t) = e^{-h^{(-1)}(x,t) + \frac{t}{2}}. \tag{88}$$

Further differentiation yields

$$\frac{u_x^2 (x,t)}{u_{xx} (x,t)} = - \frac{e^{-h^{(-1)}(x,t) + \frac{t}{2}}}{\frac{\partial}{\partial x} h^{(-1)} (x,t)}.$$

On the other hand, (42) implies

$$u_t (x,t) = -\frac{1}{2} e^{-h^{(-1)}(x,t) + \frac{t}{2}} h_x \left(h^{(-1)} (x,t), t \right) = -\frac{1}{2} \frac{e^{-h^{(-1)}(x,t) + \frac{t}{2}}}{\frac{\partial}{\partial x} h^{(-1)} (x,t)}.$$

Combining the above two equalities, we deduce that u satisfies (14).

To establish (41) and (43), we first observe that the assumption of full range yields, for each $t \geq 0$, $\lim_{x \to \pm\infty} h^{(-1)} (x,t) = \pm\infty$. Both assertions, then, follow from (88).

ii) Let u be an increasing and strictly concave function, defined for $(x,t) \in \mathbb{R} \times [0, +\infty)$ and satisfying (14), (41) and (43). Using that u_x is invertible, with $(u_x)^{(-1)} : (0, +\infty) \times [0, +\infty) \to \mathbb{R}$, we define, for $(x,t) \in \mathbb{R} \times [0, +\infty)$, the function h by

$$h (x,t) = (u_x)^{(-1)} \left(e^{-x + \frac{t}{2}}, t \right). \tag{89}$$

Note that h is invertible in the space variable since

$$h_x (x,t) = - \frac{e^{-x + \frac{t}{2}}}{u_{xx} (h (x,t), t)} > 0.$$

Differentiating (14) yields

$$u_{xt} = u_x - \frac{1}{2} \frac{u_x^2 u_{xxx}}{u_{xx}^2}. \tag{90}$$

In turn,

$$u_{xt} (x,t) = \left(-\frac{\partial}{\partial t} h^{(-1)} (x,t) + \frac{1}{2} \right) u_x (x,t),$$

$$u_{xx} (x,t) = - \left(\frac{\partial}{\partial x} h^{(-1)} (x,t) \right) u_x (x,t) \tag{91}$$

and

$$u_{xxx} (x,t) = \left(-\frac{\partial^2}{\partial x^2} h^{(-1)} (x,t) + \left(\frac{\partial}{\partial x} h^{(-1)} (x,t) \right)^2 \right) u_x (x,t).$$

Combining the above, we obtain that $h^{(-1)}$ satisfies

$$\frac{\partial}{\partial t}h^{(-1)}(x,t)+\frac{1}{2}\frac{\frac{\partial^2}{\partial x^2}h^{(-1)}(x,t)}{\left(\frac{\partial}{\partial x}h^{(-1)}(x,t)\right)^2}=0,$$

and using Lemma 27 we deduce that its spatial inverse, h, solves (19). On the other hand, (89) and (41) yield that $h(0,0)=0$. Finally, (89) and the Inada conditions yield

$$\lim_{x\to-\infty}h(x,t)=-\infty \quad \text{and} \quad \lim_{x\to+\infty}h(x,t)=+\infty.$$

Therefore, h solves (19), is strictly increasing and of full range, for each $t\geq 0$. Using part ii) of Proposition 9 we obtain (39) for some $\nu\in\mathcal{B}^{+}(\mathbb{R})$ with the appropriate properties.

It remains to show that u is given by (42). Using **(**14**)**, (89) and the form of $u_{xx}(x,t)$, we, in turn, obtain

$$u_t(x,t)=-\frac{1}{2}e^{-h^{(-1)}(x,t)+\frac{t}{2}}h_x\left(h^{(-1)}(x,t),t\right).$$

Integrating and using (41) yields

$$u(x,t)=\int_0^t u_t(x,s)\,ds+\int_0^x u_x(z,0)\,dz,$$

and (42) follows from direct integration. Note that the above two integrals are well defined as it follows from arguments used in the proof of part i). We easily conclude.

The next result will be used in the proofs that follow.

Lemma 28 *Let h be such that Range$(h)=(0,+\infty)$ (resp. Range$(h)=(-\infty,0)$. Then, for each x, $h^{(-1)}(x,t)$ is increasing (resp. decreasing) in t.*

Proof. We only look at the case *Range*$(h)=(0,+\infty)$. Using (50) and differentiating the identity $h\left(h^{(-1)}(x,t),t\right)=x$ with respect to time yields the claimed monotonicity of $h^{(-1)}(x,t)$. ■

Proof of Proposition 14: i) We first establish that u in (51) is well defined for $x>0$, $t\geq 0$. From (49) and the assumptions on the measure ν, we easily deduce that

$$h_x(x,t)=\int_{1^+}^{+\infty}e^{yx-\frac{1}{2}y^2t}\nu(dy). \tag{92}$$

Therefore,

$$\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}}h_x\left(h^{(-1)}(x,s),s\right)ds$$

$$= \int_0^t \int_{1^+}^{+\infty} e^{(y-1)h^{(-1)}(x,s)+\frac{s}{2}-\frac{1}{2}y^2 s}\nu(dy)\, ds$$
$$\leq te^{-h^{(-1)}(x,0)+\frac{t}{2}} \int_{1^+}^{+\infty} e^{yh^{(-1)}(x,t)}\nu(dy),$$

where we used Lemma 27. The finiteness of the integral then follows from the assumptions on the measure ν.

The finiteness of the second integral in (51) also follows. Indeed, first observe that (49) yields

$$\lim_{x\to-\infty} h(x,t) = 0 \quad \text{and} \quad \lim_{x\to+\infty} h(x,t) = +\infty. \tag{93}$$

Using the above, (92) and Tonelli's theorem, we obtain

$$\int_0^x e^{-h^{(-1)}(z,0)}dz = \int_{-\infty}^{h^{(-1)}(x,0)} e^{-z'} h_x(z',0)\, dz'$$
$$= \int_{-\infty}^{h^{(-1)}(x,0)} \int_{1^+}^{+\infty} e^{(y-1)z'}\nu(dy)\, dz' = \int_{1^+}^{+\infty} \int_{-\infty}^{h^{(-1)}(x,0)} e^{(y-1)z'} dz' \nu(dy)$$
$$= \int_{1^+}^{+\infty} \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - \lim_{z'\to-\infty} e^{(y-1)z'} \right) \nu(dy).$$

Using that $\lim_{z'\to-\infty} e^{(y-1)z'} = 0$ for $y > 1$ and that $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$ we deduce that

$$\int_0^x e^{-h^{(-1)}(z,0)}dz = \int_{1^+}^{+\infty} \frac{1}{y-1} e^{(y-1)h^{(-1)}(x,0)}\nu(dy). \tag{94}$$

For $\varepsilon > 0$, we then have

$$\int_0^x e^{-h^{(-1)}(z,0)}dz = \int_{1^+}^{1+\varepsilon} \frac{1}{y-1} e^{(y-1)h^{(-1)}(x,0)}\nu(dy)$$
$$+ \int_{1+\varepsilon}^{+\infty} \frac{1}{y-1} e^{(y-1)h^{(-1)}(x,0)}\nu(dy)$$
$$\leq \max\left(1, e^{\varepsilon h^{(-1)}(x,0)}\right) \int_{1^+}^{1+\varepsilon} \frac{\nu(dy)}{y-1} + \frac{e^{-h^{(-1)}(x,0)}}{\varepsilon} \int_{1+\varepsilon}^{+\infty} e^{yh^{(-1)}(x,0)}\nu(dy).$$

Using the assumptions on the measure ν we easily conclude.

The fact that u solves (14) and has the claimed monotonicity and concavity properties follows from arguments similar to the ones used in the proof of part i) in Proposition 10.

Next, we establish (52). We first show that

$$\lim_{x\to 0} \int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds = 0. \tag{95}$$

Indeed, note that the above integrand is monotone in x. This follows easily from its representation,

$$e^{-h^{(-1)}(x,t)+\frac{t}{2}}h_x\left(h^{(-1)}\left(x,t\right),t\right)=\int_{1^+}^{+\infty}e^{(y-1)h^{(-1)}(x,t)-\frac{1}{2}y^2t+\frac{t}{2}}\nu\left(dy\right),$$

combined with the monotonicity of $h^{(-1)}$. Using the monotone convergence theorem and (93), we obtain (95).

On the other hand, (94) yields

$$\lim_{x\to 0}\int_0^x e^{-h^{(-1)}(z,0)}dz=\lim_{x\to 0}\int_{1^+}^{+\infty}\frac{1}{y-1}e^{(y-1)h^{(-1)}(x,0)}\nu\left(dy\right).$$

Using the monotone convergence theorem, (93) (for $t=0$) and that $\int_{1^+}^{+\infty}\frac{\nu(dy)}{y-1}<+\infty$ we conclude.

ii) Using arguments similar to the ones in the proof of part ii) in Proposition 10, we deduce that the function h given, for $(x,t)\in R\times[0,+\infty)$, by

$$h\left(x,t\right)=\left(u_x\right)^{(-1)}\left(e^{-x+\frac{t}{2}},t\right)\tag{96}$$

is well defined and solves (19). Moreover, the assumptions on u imply that $h\left(x,t\right)\geq 0$ and $h_x\left(x,t\right)\geq 0$. Therefore, from Proposition 10, we have that there exists $\nu\in\mathcal{B}^+\left(\mathbb{R}\right)$ satisfying (48) and such that representation (49) holds. The Inada conditions (53) then yield that the normalization constant must be chosen as $C=\int_{0^+}^{+\infty}\frac{1}{y}\nu\left(dy\right)$.

Using (52) and working along similar arguments used in the proof of Proposition 10, we deduce the representation (51).

It remains to establish that ν satisfies $\nu\left((0,1]\right)=0$ and $\int_{1^+}^{+\infty}\frac{\nu(dy)}{y-1}<+\infty$. We argue by contradiction. To this end, we first note that because of (52), we have, for $x>0$,

$$u\left(x,t\right)=\int_0^x u_x\left(z,t\right)dz=\int_{-\infty}^{h^{(-1)}(x,t)}\int_0^{+\infty}e^{(y-1)z'+\frac{t}{2}\left(1-y^2\right)}\nu\left(dy\right)dz',$$

where we used (89) and (93). We, then, observe that ν cannot include a Dirac measure at $y=1$ as this would yield

$$u\left(x,t\right)\geq\int_{-\infty}^{h^{(-1)}(x,t)}dz'=+\infty,$$

contradicting the finiteness of $u\left(x,t\right)$. Therefore, we must have

$$u\left(x,t\right)=\int_{-\infty}^{h^{(-1)}(x,t)}\int_{0^+}^{1^-}e^{(y-1)z'+\frac{t}{2}\left(1-y^2\right)}\nu\left(dy\right)dz'$$
$$+\int_{-\infty}^{h^{(-1)}(x,t)}\int_{1^+}^{+\infty}e^{(y-1)z'+\frac{t}{2}\left(1-y^2\right)}\nu\left(dy\right)dz',$$

and, in turn, for $x > 0$,

$$\int_{-\infty}^{h^{(-1)}(x,t)} \int_{0^+}^{1^-} e^{(y-1)z' + \frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) dz' < +\infty \tag{97}$$

and

$$\int_{-\infty}^{h^{(-1)}(x,t)} \int_{1^+}^{+\infty} e^{(y-1)z' + \frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) dz' < +\infty. \tag{98}$$

However, using Tonelli's theorem, we deduce,

$$\begin{aligned}
&\int_{-\infty}^{h^{(-1)}(x,t)} \int_{0^+}^{1^-} e^{(y-1)z' + \frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) dz' \\
&= \int_{0^+}^{1^-} \int_{-\infty}^{h^{(-1)}(x,t)} e^{(y-1)z' + \frac{t}{2}\left(1-y^2\right)} dz' \nu\left(dy\right) \\
&= \int_{0^+}^{1^-} \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,t)} - \lim_{z' \to -\infty} e^{(y-1)z'} \right) e^{\frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right).
\end{aligned}$$

and we easily get a contradiction to (97) if $\nu\left((0,1)\right) \neq 0$.

Similarly, for all $x > 0$, we must have

$$\begin{aligned}
&\int_{-\infty}^{h^{(-1)}(x,t)} \int_{1^+}^{+\infty} e^{(y-1)z' + \frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) dz' \\
&= \int_{1^+}^{+\infty} \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,t)} - \lim_{z \to -\infty} e^{(y-1)z} \right) e^{\frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) \\
&= \int_{1^+}^{+\infty} \frac{1}{y-1} e^{(y-1)h^{(-1)}(x,t) + \frac{t}{2}\left(1-y^2\right)} \nu\left(dy\right) < +\infty.
\end{aligned}$$

By assumption, for each $t \geq 0$, $Range(h^{(-1)}) = (-\infty, +\infty)$. Therefore, for $t = 0$, there exists $x_0(0)$ such that $h^{(-1)}\left(x_0\left(0\right), 0\right) = 0$. We easily conclude.

Proof of Proposition 15: We only prove some of the main points, for the rest of the proof follows along similar arguments as in the previous proof. To this end, we first show that the function given in (54) is well defined. Indeed,

$$\begin{aligned}
&\int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x\left(h^{(-1)}\left(x,s\right), s\right) ds \\
&= \int_0^t \int_{0+}^{+\infty} e^{(y-1)h^{(-1)}(x,s) + \frac{s}{2} - \frac{1}{2}y^2 s} \nu\left(dy\right) ds \\
&\leq t e^{-h^{(-1)}(x,0) + \frac{t}{2}} \int_{0^+}^{+\infty} e^{y h^{(-1)}(x,t)} \nu\left(dy\right),
\end{aligned}$$

where we used Lemma 27. The finiteness of the integral follows from the assumptions on the measure ν.

Moreover, for $x > x_0$ (the case $x < x_0$ follows similarly),

$$\int_{x_0}^{x} e^{-h^{(-1)}(z,0)} dz = \int_{h^{(-1)}(x_0,0)}^{h^{(-1)}(x,0)} e^{-z'} h_x(z',0)\, dz'$$

$$= \int_{h^{(-1)}(x_0,0)}^{h^{(-1)}(x,0)} \int_{0+}^{+\infty} e^{(y-1)z'} \nu(dy)\, dz' = \int_{0+}^{+\infty} \int_{h^{(-1)}(x_0,0)}^{h^{(-1)}(x,0)} e^{(y-1)z'} dz' \nu(dy)$$

$$= \int_{0+}^{+\infty} \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \nu(dy).$$

We have

$$\int_{0+}^{+\infty} \left| \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \right| \nu(dy)$$

$$= \int_{0+}^{2} \left| \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \right| \nu(dy)$$

$$+ \int_{2}^{+\infty} \left| \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \right| \nu(dy)$$

$$\leq \int_{0+}^{2} \left| \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \right| \nu(dy)$$

$$+ e^{-h^{(-1)}(x,0)} \int_{2}^{+\infty} e^{yh^{(-1)}(x,0)} \nu(dy) + e^{-h^{(-1)}(x_0,0)} \int_{2}^{+\infty} e^{yh^{(-1)}(x_0,0)} \nu(dy).$$

On the other hand,

$$\frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right)$$

$$= \frac{1}{y-1} \left(\sum_{n=1}^{+\infty} \left(\frac{(y-1)^n \left(h^{(-1)}(x,0)\right)^n}{n!} - \frac{(y-1)^n \left(h^{(-1)}(x_0,0)\right)^n}{n!} \right) \right)$$

$$= \sum_{n=1}^{+\infty} \left(\frac{(y-1)^{n-1} \left(h^{(-1)}(x,0)\right)^n}{n!} - \frac{(y-1)^{n-1} \left(h^{(-1)}(x_0,0)\right)^n}{n!} \right).$$

For $0 \leq y \leq 2$,

$$\left| \frac{1}{y-1} \left(e^{(y-1)h^{(-1)}(x,0)} - e^{(y-1)h^{(-1)}(x_0,0)} \right) \right|$$

$$\leq \sum_{n=1}^{+\infty} \left(\frac{|y-1|^{n-1} \left|h^{(-1)}(x,0)\right|^n}{n!} + \frac{|y-1|^{n-1} \left|h^{(-1)}(x_0,0)\right|^n}{n!} \right)$$

$$\leq e^{\left|h^{(-1)}(x,0)\right|} + e^{\left|h^{(-1)}(x,0)\right|} - 2.$$

Combining the above yields

$$\int_{0+}^{+\infty}\left|\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x,0)}-e^{(y-1)h^{(-1)}(x_0,0)}\right)\right|\nu\left(dy\right)$$

$$\leq\left(e^{\left|h^{(-1)}(x,0)\right|}+e^{\left|h^{(-1)}(x,0)\right|}-2\right)\nu\left([0,2]\right)$$

$$+e^{-h^{(-1)}(x,0)}\int_{2}^{+\infty}e^{yh^{(-1)}(x,0)}\nu\left(dy\right)+e^{-h^{(-1)}(x_0,0)}\int_{2}^{+\infty}e^{yh^{(-1)}(x_0,0)}\nu\left(dy\right)$$

and we easily conclude.

Next we show that under the assumptions on the measure ν, (55) holds.

First we assume that $\nu\left((0,1]\right)>0$. Observe that for x sufficiently small,

$$\int_{x_0}^{x}e^{-h^{(-1)}(z,0)}dz=-\int_{0+}^{+\infty}\int_{h^{(-1)}(x,0)}^{h^{(-1)}(x_0,0)}e^{(y-1)z'}dz'\nu\left(dy\right)$$

$$=-\left(\int_{0+}^{1}\int_{h^{(-1)}(x,0)}^{h^{(-1)}(x_0,0)}e^{(y-1)z'}dz'\nu\left(dy\right)+\int_{1^+}^{+\infty}\int_{h^{(-1)}(x,0)}^{h^{(-1)}(x_0,0)}e^{(y-1)z'}dz'\nu\left(dy\right)\right)$$

$$\leq-\int_{0+}^{1}\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x_0,0)}-e^{(y-1)h^{(-1)}(x,0)}\right)\nu\left(dy\right).$$

Passing to the limit and using the monotone convergence theorem yields

$$\lim_{x\to0}\int_{0+}^{1}\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x_0,0)}-e^{(y-1)h^{(-1)}(x,0)}\right)\nu\left(dy\right)=+\infty.$$

Next we look at the case, $\nu\left((0,1]\right)=0$ and $\int_{1+}^{+\infty}\frac{\nu(dy)}{y-1}=+\infty$. We then have

$$\int_{x_0}^{x}e^{-h^{(-1)}(z,0)}dz=-\int_{1^+}^{+\infty}\int_{h^{(-1)}(x,0)}^{h^{(-1)}(x_0,0)}e^{(y-1)z'}dz'\nu\left(dy\right)$$

$$=-\int_{1^+}^{+\infty}\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x_0,0)}-e^{(y-1)h^{(-1)}(x,0)}\right)\nu\left(dy\right).$$

Using the monotone convergence theorem and that

$$\lim_{x\to0}\left(e^{(y-1)h^{(-1)}(x_0,0)}-e^{(y-1)h^{(-1)}(x,0)}\right)=e^{(y-1)h^{(-1)}(x_0,0)}$$

yields

$$\lim_{x\to0}\int_{x_0}^{x}e^{-h^{(-1)}(z,0)}dz=-\int_{1^+}^{+\infty}\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x_0,0)}\right)\nu\left(dy\right).$$

The elementary inequality $e^{(y-1)h^{(-1)}(x_0,0)}\geq1+(y-1)\,h^{(-1)}\left(x_0,0\right)$ in turn implies

$$-\int_{1^+}^{+\infty}\frac{1}{y-1}\left(e^{(y-1)h^{(-1)}(x_0,0)}\right)\nu\left(dy\right)$$

$$\leq -\int_{1^+}^{+\infty} \frac{1}{y-1}\left(1+(y-1)\,h^{(-1)}(x_0,0)\right)\nu(dy)$$

$$= -h^{(-1)}(x_0,0)\int_{1^+}^{+\infty} \frac{1}{y-1}\nu(dy) - h^{(-1)}(x_0,0)\,\nu((1,+\infty)),$$

and we easily conclude.

References

[1] Barrier F., Rogers L.C. and M. Tehranchi: A characterization of forward utility functions, preprint, (2009).

[2] Goldstein, D.G., Johnson E.J. and W.F. Sharpe: Measuring consumer risk-return trade-offs, preprint (2006).

[3] Hu, Y. and X.Y. Zhou: Constrained stochastic LQ control with random coefficients, and application to mean–variance portfolio selection, *SIAM Journal on Control and Optimization*, 44, 444-466 (2005).

[4] Jin, H. and X.Y. Zhou: Behavioral portfolio selection in continuous time, *Mathematical Finance*, 18, 385-426 (2008).

[5] Karatzas, I. and G. Zitkovic: Optimal consumption from investment and random endowment in incomplete semimartingale markets, *Annals of Applied Probability,* 31(4), 1821-1858 (2003).

[6] Li, X. and X.Y. Zhou: Continuous-time mean–variance efficiency: The 80% rule", *Annals of Applied Probability*, 16, 1751-1763 (2006).

[7] Lipster, R.S. and A.N. Shiryaev: *Statistics of Random Processes*, *II*: *Applications* (2nd edition), Springer, Berlin (2000).

[8] Merton, R. : Lifetime portfolio selection under uncertainty: the continuous-time case, *The Review of Economics and Statistics,* 51, 247-257 (1969).

[9] Musiela, M. and T. Zariphopoulou: Optimal asset allocation under forward exponential criteria, *Markov Processes and Related Topics: A Festschrift for Thomas. G. Kurtz,* IMS Collections, Institute of Mathematical Statistics, 4, 285-300 (2008).

[10] Musiela, M. and T. Zariphopoulou: Portfolio choice under dynamic investment performance criteria, *Quantitative Finance,* 9(2), 161-170 (2009).

[11] Musiela, M. and T. Zariphopoulou: Stochastic partial differential equations in portfolio choice, submitted for publication (2009).

[12] Musiela, M. and T. Zariphopoulou: Investments and forward utilities, preprint (2006).

[13] Penrose, R.: A generalized inverse for matrices, *Proceedings of Cambridge Philosophical Society,* 51, 406-413 (1955).

[14] Schachermayer, W.: A super-martingale property of the optimal portfolio process, *Finance and Stochastics*, 7, 433-456 (2003).

[15] Sharpe W.F., Goldstein D.G. and P.W. Blythe: The distribution builder: A tool for inferring investor's preferences, preprint (2000).

[16] Sharpe W.F.: Expected utility asset allocation, preprint (2006).

[17] Widder, D.V.: *The heat equation*, Academic Press (1975).

[18] Zariphopoulou, T. and T. Zhou: Investment performance measurement under asymptotically linear local risk tolerance, Mathematical Modeling and Numerical Methods in Finance, *Handbook of Numerical Analysis*, A. Bensoussan and Q. Zhang (eds.), North Holland, 227-254 (2009).

[19] Zitkovic, G.: A dual characterization of self-generation and log-affine forward performances, *Annals of Applied Probability,* to appear (2008).