Initial investment choice and optimal future allocations*

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Abstract

The paper offers a new perspective on optimal portfolio choice by investigating how knowledge of an investor’s desirable initial investment choice can be used to determine his future optimal portfolio allocations. Optimality of investment decisions is built on forward investment performance criteria. The analysis uses the connection between a fast diffusion equation, satisfied by the local risk tolerance, and the backward heat equation. Complete solutions for the case of monotone performance criteria are provided as well as various examples.

1 Introduction

The powerful and elegant theory of expected utility yields optimal portfolio choices for risk averse investors in stochastic market environments. In the classical continuous time framework, introduced in [5], the investor’s preferences are assigned at the end of the trading horizon. The optimal investment strategy is, then, the one that delivers the maximal expected (indirect) utility of terminal wealth. Recently, the authors proposed an alternative approach to optimal portfolio choice which is based on the so-called forward investment performance criterion (see, among others, [8]). In this approach, the investor does not choose her risk preferences at a single point in time, as it is the case in the Merton model, but has the flexibility to revise them dynamically.

From the methodological point of view, one of the fundamental differences between the two approaches is that the investor specifies her utility at the beginning of the trading horizon and not at the end. Whether one works in the traditional or the alternative framework, the specification of the terminal, or initial,

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risk preferences remains a challenging task. In the context of forward portfolio choice, this question was recently examined in [6]. Therein, one shows how the investor’s preferences may be inferred from desirable distributional properties of future investment targets. An important consequence of this methodology is that the individual risk preferences are calibrated to the specific market environment and the investment opportunities this environment offers. This is in contrast to the classical framework in which the utility function is assigned exogenously, in isolation to the market in which the investor operates. Inference of risk preferences from similar probabilistic criteria has been also proposed and analyzed in a static model by W. Sharpe and his coauthors in [3], [4], [11] and [12].

Herein, we provide an additional point on view on portfolio choice and the specification of the investor’s risk preferences. The analysis is built on an entirely different perspective which is, from one hand, quite unorthodox with respect to the classical academic ideas but, on the other, it appears to be much closer to the rationale applied in investment practice. Specifically, instead of requiring that the investor knows his risk preferences at the beginning of the investment horizon, it is assumed that he is able to specify the portfolio allocations that are desirable to him at initiation. The motivation to consider this direction comes mainly from the fact that the concept of utility is quite elusive and difficult to quantify. The inspiration comes from a paper written by F. Black ([2]) on similar ideas. One reads:

....“Utility” is a foreign concept for most individuals, and the traditional way of mapping an individual’s utility function is to ask him about his preferences among a number of gambles.

Instead of stating his preferences among various gambles, the individual can specify his consumption function and his initial investment function. He can give the rate at which he would consume his wealth at various times in the future as a function of the amount of his wealth at those times. And he can tell how much he would invest currently in the market portfolio as a function of his current wealth.

It is possible to eliminate the individual’s utility function entirely from these equations, giving a set of equations involving only his investment function and his consumption function....

Therein an infinitesimal analysis is proposed for the specification of the future optimal investment and consumption policies once their initial values are chosen. The case when these initial choices are proportional to the investor’s wealth is solved in detail.

In this paper, we build on the same ideas. We provide a complete solution for arbitrary initial investment choices. We do not allow for intermediate consumption and, moreover, focus on forward investment performance processes that are monotone in time. We show how knowledge of the initial desired allocation can

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be used to recover all future portfolio processes, the associated wealth process and, finally, the investment criterion with respect to which they (portfolio and wealth) are optimal.

We note that the entire analysis bypasses the need to know the risk preferences. Indeed, we will see that risk preferences are identified at the end when one seeks the criterion with respect to which the already constructed processes are optimal. The solution approach is based on a deep connection between the so-called local risk tolerance function and harmonic functions as well on the specification of a measure whose bilateral Laplace transform yields the latter. This measure becomes the defining element for all quantities of interest. We show how the initial desired allocations reveals it. A byproduct of this analysis is the specification of the set of initial allocations that lead to well defined future investment choices. One can say that this is the set of feasible initial investment choices.

The paper is organized as follows. In sections 2 and 3 we introduce the model and pose the portfolio choice problem. In section 4 we provide auxiliary results on the representation of the local risk tolerance function and its connection with harmonic solutions. In section 5 we provide the main results and answers to the questions in consideration. We conclude with various examples in section 6.

2 The model and the investment performance criterion

The market environment consists of a riskless bond and a stock. The stock price, $S_t$, $t \geq 0$, is modelled as an Itô process following

$$dS_t = S_t (\mu_t dt + \sigma_t dW_t),$$  \hspace{1cm} (1)

with $S_0 > 0$. The process $W_t$ is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients $\mu_t$ and $\sigma_t$, $t \geq 0$, are $\mathcal{F}_t$-adapted processes. The riskless asset, the savings account, has the price process $B_t$, $t \geq 0$, satisfying

$$dB_t = r_t B_t dt,$$

with $B_0 = 1$, and for a nonnegative, $\mathcal{F}_t$-adapted interest rate process $r_t$. The market coefficients $\mu_t$, $\sigma_t$ and $r_t$ are taken to satisfy the appropriate integrability conditions.

We define the process $\lambda_t$, $t \geq 0$, by

$$\lambda_t = \frac{\mu_t - r_t}{\sigma_t},$$  \hspace{1cm} (2)

and we throughout assume that it is bounded by a deterministic constant $c > 0$, i.e., for all $t \geq 0$, $|\lambda_t| \leq c$. We will be occasionally referring to this process as the market price of risk.
Starting at \( t = 0 \) with an initial endowment \( x \in \mathbb{R} \), the investor invests, at any time \( t > 0 \), in the two assets. The present value of the amounts invested in the bond and the stock are denoted, respectively, by \( \pi_t^0 \) and \( \pi_t \).

The present value of her current total investment is, then, given by \( \pi_t^0 + \pi_t, \quad t \geq 0 \). Using (1) we have that it satisfies
\[
\frac{d\pi_t^0}{\pi_t} = \sigma_t \lambda t \, dt + dW_t,
\]
with \( \lambda = \pi_t^0 \). The investment policies \( \pi_t, \quad t \geq 0 \), will play the role of control processes and are taken to satisfy the usual assumptions of being self-financing, \( \mathcal{F}_t \)-progressively measurable and satisfying the integrability condition \( E \left( \int_0^t |\sigma_s \pi_s|^2 \, ds \right) < \infty, \quad t > 0 \). We denote the set of admissible strategies by \( \mathcal{A} \).

Recently, a selection criterion for investment strategies in \( \mathcal{A} \) was introduced in [8] (see, also, [10]). It is represented by a process, the so-called forward investment performance. This criterion is defined for all times and complements the traditional one which is based on the maximal expected utility of terminal wealth (value function process), defined only on \([0, T], \quad T < +\infty \).

As the definition below states, a strategy is deemed optimal if it generates a wealth process whose average performance is maintained over time. In other words, the average performance of this strategy at any future date, conditional on today’s information, preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is, then, sub-optimal. The intuition behind these requirements comes from the analogous properties of the value function process, namely, its martingality along optimal policies and its supermartingality away from the optimum. We refer the reader to [8] for a detailed discussion on the new approach.

**Definition 1** An \( \mathcal{F}_t \)-adapted process \( U (x, t) \) is a forward investment performance if for \( t \geq 0 \) and \( x \in \mathbb{R} \):

i) the mapping \( x \rightarrow U (x, t) \) is strictly concave and increasing,

ii) for each \( \pi \in \mathcal{A} \), \( E (U (X^\pi_s, t)) < \infty \), and

\[
E \left( U (X^\pi_s, s) \mid \mathcal{F}_t \right) \leq U (X^\pi_t, t), \quad s \geq t,
\]

iii) there exists \( \pi^*_t \in \mathcal{A} \), for which

\[
E \left( U (X^\pi^*_s, s) \mid \mathcal{F}_t \right) = U (X^\pi^*_t, t), \quad s \geq t.
\]

**Remark 2** Intuitively speaking, \( U (x, t) \) represents the stochastic utility that is generated by the policy of investing all wealth in the riskless asset across time. Note that even though this is a riskless strategy, it generates nondeterministic utility, for it is implemented in a stochastic environment. Note, also, that this interpretation is directly aligned with the fact that the bond has been chosen as the numeraire.
Characterizing the class of processes which satisfy the above definition is a challenging problem. An important result in this direction, introduced in [9], is a stochastic partial differential equation for the investment performance process, namely,

\[
dU(x,t) = \frac{1}{2} \left( \frac{t}{U_{xx}(x,t)} \right)^2 dt + a(x,t) dW_t,
\]

where \(a(x,t)\) is an \(\mathcal{F}_t\)-adapted process and \(\lambda_t, t \geq 0\), as in (2).

The volatility process \(a(x,t)\) is the novel element of the forward approach in portfolio choice. This input is up to the investor to choose, in contrast to the classical formulation in which the volatility coefficient of the value function process is uniquely determined. We refer the reader to [9] for further discussion on the subject; see, also, [1] and [16] for other recent developments in this new approach.

In this paper, we do not examine issues of existence, uniqueness and regularity of the investment performance process. Rather, we investigate how information about preferred personal portfolio choice can be used to build a framework for investment advice across time.

3 The portfolio selection problem

We consider an individual who starts today with some initial wealth and faces investment decisions in upcoming times. The investor specifies his initial desired investment allocation between the two accounts.

We are interested in the following questions:

- How does the investor’s initial investment choice affect, and to what extent determine his future allocations?
- Is there an investment criterion that is consistent with these policies, in that these portfolios - initial and future - are optimal with respect to it?
- Are all initial investment choices admissible in the sense that they lead to feasible future investment allocations and, in general, to a well-posed portfolio choice problem?

We will study these questions focusing on the class of investors who have time-monotone investment criteria, specifically, when the process \(U(x,t)\) is time-decreasing. This is the simplest class of forward investment performance processes which, nevertheless, offers rich intuition and valuable insights. It is easy to see that this case corresponds the zero volatility choice, \(a(x,t) = 0, t \geq 0\), in the performance SPDE (6).

To ease the exposition, we first provide some auxiliary results on monotone investment performance processes. These results can be found in [8] and are presented herein without a proof. To this end, we recall that such processes
are represented as a compilation of market related input with a deterministic function of space and time. Specifically, for \( t \geq 0 \), \( U(x, t) \) is given by

\[
U(x, t) = u(x, A_t),
\]

(7)

where \( u(x, t) \) is increasing and strictly concave in \( x \), and satisfies the fully nonlinear equation

\[
u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}.
\]

(8)

The process \( A_t \) depends only on market movements up to current time \( t \) and is given by

\[
A_t = \int_0^t \lambda_s^2 ds
\]

(9)

with \( \lambda_s \), \( t \geq 0 \), as in (2).

In [8] it was also shown that the investor’s optimal wealth and optimal portfolio processes, denoted respectively by \( X^*_t \) and \( \pi^*_t \), \( t \geq 0 \), satisfy an autonomous system (cf. (13)) of stochastic differential equations whose coefficients depend functionally on the spatial derivatives of \( u \). Specifically its coefficients can be expressed in terms of the so-called local risk tolerance function defined by

\[
r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}.
\]

(10)

Another important result therein is that the above function satisfies an autonomous equation (cf. (11)). The autonomous system and equation will play pivotal role in our analysis as explained in the sequel.

The results below can be found in Propositions 6 and 9 in [8]. Because later on we will be working with different cases of the spatial domain, we use, for the moment, the generic notation \( D \).

**Theorem 3** i) Let \( u(x, t) \) solve (8) and \( r(x, t) \) be as in (10). Then, for \( (x, t) \in D \times [0, +\infty) \), \( r(x, t) \) solves

\[
r_t + \frac{1}{2} r^2 r_{xx} = 0.
\]

(11)

ii) Define the process

\[
R^*_t = r(X^*_t, A_t),
\]

(12)

with \( A_t, t \geq 0 \), as in (9) and consider the system

\[
\begin{cases}
    dX^*_t = r(X^*_t, A_t) \lambda_t (\lambda_t dt + dW_t) \\
    dR^*_t = r_x(X^*_t, A_t) dX^*_t,
\end{cases}
\]

(13)

with \( X^*_0 = x \) and \( R^*_0 = r(x, 0) \). Assume that \( X^*_t \) and \( R^*_t \) solve (13) and define \( \pi^*_t \), \( t \geq 0 \), by

\[
\pi^*_t = \frac{\lambda_t}{\sigma_t} R^*_t.
\]

(14)

Then \( \pi^*_t \) generates \( X^*_t \) and, moreover, is optimal under the criterion \( U(x, t) \) given in (7).
Returning to the problem we want to analyze herein, let us assume that the investor specifies her initial desired allocation as functions of her wealth, say \( \pi_0^*(x) \) and \( x - \pi_0^*(x) \), for each wealth level \( x \in \mathbb{D} \). The questions posed at the beginning of the section can be, then, formulated as follows:

- How does the form of \( \pi_0^*(x) \) affect and, to what extent, determine the future allocation processes \( \pi^*_t, t > 0 \)?
- How can one infer the investor’s performance criterion, \( u(x, A_t), t \geq 0 \), with respect to which the policies \( \pi^*_t, t \geq 0 \), are optimal?
- What is the class of admissible initial allocations \( \pi_0^*(x) \)?

Theorem 3 gives valuable insights which will be central in our solution approach. Firstly, note that solving the above system essentially bypasses the need to specify the investment criterion. Indeed, knowing the function \( r(x, t) \) and solving (13) yields directly the optimal processes \( X^*_t \) and \( \pi^*_t \). This is in contrast to the classical setting in which one starts with the specification of terminal utility, then finds the value function process and, in turn, recovers the optimal policies.

We will utilize the results of the above theorem as follows. Firstly note that the investor’s initial preferred allocation \( \pi_0^*(x) \) yields the initial condition \( r(x, 0) \), as (12) and (14) indicate. The function \( r(x, t) \) together with equation (11), specifies \( r(x, t), t > 0 \). The latter and its spatial derivatives give the functional coefficients in (13) which, in turn, yield the processes \( X^*_t \) and \( R^*_t \). The optimal allocation and the associated investment criterion, \( \pi^*_t \) and \( U(x, t) \), are, then, recovered from (14) and (7), respectively.

In order to execute these steps, one needs to solve equation (11) and the system (13), and to construct the function \( u \). We discuss these problems next. We start with auxiliary results for solutions to (11) and (8).

### 4 Analysis

We focus on specific classes of solutions to equations (11) and (8), namely, positive solutions of (11) and strictly concave and increasing solutions of (8). We stress that, for the moment, we do not assume that these solutions are related to each other as (34) might indicate (as well as the titles of the subsections below). This connection will be discussed at the end of the section. For now we treat the two equations separately.

We recall the backward heat equation,

\[
 h_t + \frac{1}{2} h_{xx} = 0. \tag{15}
\]

We consider strictly increasing solutions to the above equation and construct the associated solutions to (11) and (8) by appropriate nonlinear and integral transformations (see Propositions 4 and 8). The motivation to involve (15)
comes from the fact that its harmonic solutions have very explicit, and quite useful for the investment problem at hand, representations. These representation results are also discussed in this section.

As in the previous section, we use $\mathbb{D}$ to denote the generic domain of the function $r$.

### 4.1 Local risk tolerance and harmonic functions

We start with a result that relates solutions of (15) and (11).

**Proposition 4** Let $h : \mathbb{R} \times [0, +\infty) \to \mathbb{D}$ be a strictly increasing solution to (15) and define $r : \mathbb{D} \times [0, +\infty) \to \mathbb{R}^+$ by

$$r(x,t) = h_x \left( h^{-1}(x,t), t \right). \tag{16}$$

Then $r$ solves (11).

**Proof.** Differentiating (15) yields

$$h_t(x,t) r_x(h(x,t), t) + r_t(h(x,t), t) = h_{xt}(x,t)$$

and

$$h_{xx}(x,t) r_x(h(x,t), t) + h_x^2(x,t) r_{xx}(h(x,t), t) = h_{xxx}(x,t).$$

Using (15) we deduce

$$h_{xt}(x,t) + \frac{1}{2} h_{xxx}(x,t) = r_x(h(x,t), t) \left( h_t(x,t) + \frac{1}{2} h_{xx}(x,t) \right)$$

$$= r_t(h(x,t), t) + \frac{1}{2} h_x^2(x,t) r_{xx}(h(x,t), t).$$

Using (15) once more we have

$$r_t(h(x,t), t) + \frac{1}{2} h_x^2(x,t) r_{xx}(h(x,t), t) = 0,$$

and we easily conclude. □

**Remark 5** In portfolio choice, a quantity that occasionally appears is the reciprocal of the local risk tolerance that is called, in analogy, the local risk aversion,

$$\gamma(x,t) = \frac{1}{r(x,t)}.$$ 

Direct differentiation of $\gamma(x,t)$ and use of (11) yield that $\gamma(x,t)$ satisfies the equation

$$\gamma_t + \frac{1}{2} (F(\gamma))_{xx} = 0 \quad \text{with} \ F(\gamma) = \gamma^{-1}. \tag{17}$$

Equation (11) is often referred to as a fast diffusion equation while equation (17) is a porous medium equation. We refer the reader to [13] for an extensive study on these nonlinear equations. We note, however, that (11) and (17) correspond to cases in which the "critical exponents" are beyond the range in which solutions are well defined for all times.
4.2 Representation of strictly increasing harmonic functions

We continue with representation results for strictly increasing solutions to (15). They appeared recently in [6] and we refer the reader therein for the related proofs. Here, we only highlight the main findings and representative cases.

The key idea in constructing monotone solutions to (15) is to use the classical representation result of Widder for positive harmonic functions. This result, recalled below (see, also, [14]), states that every positive solution of (15) can be represented in terms of the bilateral Laplace transform of a finite positive Borel measure, denoted hereafter by \( \nu \).

Widder’s theorem is not directly applicable in the portfolio problem we study because the solutions \( h \) we consider might not be positive. Indeed, as we will see in the next section, these solutions will be ultimately used to construct the investor’s optimal wealth which is not necessarily positive (see, for instance, examples 6.1). However, Widder’s theorem can be applied to the spatial derivative \( h_x \) which is a positive harmonic solution, as it follows from direct differentiation of (15) and the monotonicity of \( h \).

We introduce the following set of measures,

\[
B^+ (\mathbb{R}) = \left\{ \nu \in \mathcal{B} (\mathbb{R}) : \forall B \in \mathcal{B}, \, \nu (B) \geq 0 \text{ and } \int_{\mathbb{R}} e^{yx} \nu (dy) < \infty, \, x \in \mathbb{R} \right\} .
\]

Theorem 6 (Widder). Let \( g(x, t), \, (x, t) \in \mathbb{R} \times [0, +\infty) \), be a positive solution of (15). Then, there exists \( \mu \in B^+ (\mathbb{R}) \) such that \( g \) is represented as

\[
 g (x, t) = \int_{\mathbb{R}} e^{yx} - \frac{1}{2} y^2 t \mu (dy) .
\]

As the results in the sequel show, there is an interesting interplay between the support and the integrability properties of the measure \( \nu \), and the range of the solution \( h \). We will see that, in turn, this interplay affects the range of the local risk tolerance, the range and the limiting behavior of the differential input and, finally, the range of the investor’s optimal wealth process. We stress that \( h \) is defined, throughout, for all \( (x, t) \in \mathbb{R} \times [0, +\infty) \). It is the range of \( h \) that varies as the underlying measure changes.

To this end, we introduce the following subsets of \( B^+ (\mathbb{R}) \),

\[
B_0^+ (\mathbb{R}) = \left\{ \nu \in B^+ (\mathbb{R}) \text{ and } \nu \{0\} = 0 \right\} ,
\]

\[
B^+_0 (\mathbb{R}) = \left\{ \nu \in B_0^+ (\mathbb{R}) : \nu \{(-\infty, 0)\} = 0 \right\}
\]

and

\[
B^+_0 (\mathbb{R}) = \left\{ \nu \in B_0^+ (\mathbb{R}) : \nu \{(0, +\infty)\} = 0 \right\} .
\]

It is throughout assumed that the trivial case \( \nu (\mathbb{R}) = 0 \) is excluded.
Proposition 7  i) Let $\nu \in B_+^+(\mathbb{R})$. Then, the function $h$ defined for $(x, t) \in \mathbb{R} \times [0, +\infty)$ by

$$h(x, t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2t} \frac{-1}{y} \nu(dy) \quad (20)$$

is a strictly increasing solution to (15). We have the following cases:

If $\nu(\{0\}) > 0$, or $\nu \in B_0^+(\mathbb{R})$, or $\nu \in B_+^+(\mathbb{R})$ and $\int_{0}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, or $\nu \in B_+^-(\mathbb{R})$ and $\int_{-\infty}^{0} \frac{\nu(dy)}{y} = -\infty$, then $\text{Range}(h) = (-\infty, +\infty)$, for $t \geq 0$.

If $\nu \in B_+^+(\mathbb{R})$ with $\int_{0}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ then $\text{Range}(h) = (0, +\infty)$, for $t \geq 0$, and $h$ can be written as

$$h(x, t) = \int_{0}^{\infty} e^{yx - \frac{1}{2}y^2t} \mu(dy) \quad (21)$$

with $\mu(dy) = \frac{\nu(dy)}{y}$.

If $\nu \in B_+^+(\mathbb{R})$ with $\int_{-\infty}^{0} \frac{\nu(dy)}{y} > -\infty$ then $\text{Range}(h) = (-\infty, 0)$, for $t \geq 0$, and $h$ can be written as

$$h(x, t) = \int_{-\infty}^{0} e^{yx - \frac{1}{2}y^2t} \mu(dy) \quad (22)$$

with $\mu(dy) = \frac{\nu(dy)}{y}$.

ii) Conversely, let $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be a strictly increasing solution to (15). Then, there exists $\nu \in B_+^+(\mathbb{R})$ such that $h$ is given by (20).

Moreover, if $\text{Range}(h) = (-\infty, +\infty)$, $t \geq 0$, then it must be either that $\nu(\{0\}) > 0$, or $\nu \in B_0^+(\mathbb{R})$, or $\nu \in B_+^+(\mathbb{R})$ and $\int_{0}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, or $\nu \in B_+^-(\mathbb{R})$ and $\int_{-\infty}^{0} \frac{\nu(dy)}{y} = -\infty$.

On the other hand, if $\text{Range}(h) = (0, +\infty)$ (resp. $\text{Range}(h) = (-\infty, 0)$), $t \geq 0$ and $x_0 \in \mathbb{R}$, then it must be that $\nu \in B_+^+(\mathbb{R})$ with $\int_{0}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ (resp. $\nu \in B_+^-(\mathbb{R})$ with $\int_{-\infty}^{0} \frac{\nu(dy)}{y} > -\infty$).

The proof of the above result can be found in Proposition 9 in [6]. Therein, one shows that the converse of the above statements is also true, namely, if $h$ is a strictly increasing solution of (15) then, depending on its range, there is a measure with the appropriate support which can be used for its integral representation.

4.3 Differential input and harmonic functions

Next, we discuss how strictly concave and increasing solutions of (8) can be constructed from strictly increasing solutions to (15). We stress that the harmonic function appearing in (23) (as well as in (24) and (26)) is not necessarily the same with the one in subsection 4.2. We use, with a slight abuse, the same notation only for simplicity.

In order to facilitate the exposition, we compress some of the different cases and assumptions on the measure. We also omit the case $\text{Range}(h) = (-\infty, 0)$. 


Proposition 8 Let $h$ be a strictly increasing harmonic function and $\nu$ its associated measure (cf. (20) and (21)). We have the following cases:

i) Let $\nu$ be such that $\text{Range}(h) = (-\infty, +\infty)$. Then, $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ defined by

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{-1}(x,s)} + \frac{z}{2} h(x,s) \, ds + \int_0^x e^{-h^{-1}(z,0)} \, dz,$$

with $h$ given in (20), is strictly increasing and concave in its spatial argument and satisfies equation (8).

ii) Let $\nu$ be such that $\text{Range}(h) = (0, +\infty)$ and satisfying, in addition, $\nu((0,1]) = 0$ and $\int_1^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$. Then $u : \mathbb{R}^+ \times [0, +\infty) \to \mathbb{R}$ is given by

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{-1}(x,s)} + \frac{z}{2} h(x,s) \, ds + \int_0^x e^{-h^{-1}(z,0)} \, dz,$$

and satisfies

$$\lim_{x \to 0} u(x,t) = 0, \quad \text{for } t \geq 0. \quad (25)$$

If, on the other hand, $\nu((0,1]) > 0$ and/or $\int_1^{+\infty} \frac{\nu(dy)}{y-1} = +\infty$, we have

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{-1}(x,s)} + \frac{z}{2} h(x,s) \, ds + \int_0^x e^{-h^{-1}(z,0)} \, dz,$$

for $x_0 > 0$, with

$$\lim_{x \to x_0} u(x,t) = -\infty, \quad \text{for } t \geq 0. \quad (27)$$

For each $t \geq 0$, the Inada conditions

$$\lim_{x \to -\infty} u_x(x,t) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} u_x(x,t) = 0$$

and

$$\lim_{x \to -\infty} u_x(x,t) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} u_x(x,t) = 0 \quad (28)$$

are satisfied for (23), and both (24), respectively.

In [6] one also shows that the converse of the above statements is also true, namely, if $u$ is a strictly increasing and concave solution to (8) (and certain limiting properties) then, there exists an associated solution $h$ to (15) with the appropriate representation characteristics (see Propositions 10, 14 and 15 in [6]).

4.4 Local risk tolerance and differential input

So far we have seen that solutions to (11) and (8) are directly related and can be constructed by harmonic solutions (see (16) and (23)). How are, then, $r(x,t)$
and \( u(x,t) \) related to each other if they "correspond" to the same harmonic function?

Given that the measure \( \nu \) is the defining element of \( h(x,t) \) it is immediate that \( r(x,t) \) and \( u(x,t) \) must be associated with the same measure. This, in turn, implies that the initial conditions \( r(x,0) = h(h^{-1}(x,0),0) \) and \( u(x,0) = \int_0^x e^{-h^{-1}(z,0)}dz \) must be compatible. Direct differentiation yields that

\[
-\frac{u_x(x,0)}{u_{xx}(x,0)} = r(x,0). \tag{29}
\]

Further differentiation in (41) and use of (39) yields that the above relation is also propagated to all positive times, namely, for \( t > 0 \),

\[
-\frac{u_x(x,t)}{u_{xx}(x,t)} = r(x,t). \tag{30}
\]

5 Synthesis

We are now ready to address the questions posed in section 3. We introduce the process \( M_t, t \geq 0 \),

\[
M_t = \int_0^t \lambda_s dW_s \tag{31}
\]

with \( \lambda_t \) as in (2). We also introduce a stricter than (18) assumption on the measure \( \nu \), specifically,

\[
\int_{\mathbb{R}} e^{\|x\|_2 + \frac{1}{2}y^2} t \nu(dy) < \infty, \quad t \geq 0. \tag{32}
\]

This condition is not needed for the representation results of the previous section but it is crucial in establishing the appropriate integrability, and in turn admissibility, of the candidate optimal investment process.

We start with the case when the investor’s initial wealth is unconstrained, \( x \in \mathbb{R} \). For the reader’s convenience, we rewrite some of the formulae derived earlier. On the other hand, we do not write explicitly the conditions on the measure that yield full range of the solution \( h \), for they are already listed in the first part of Proposition 7.

Theorem 9 Let \( \pi^*_0 : \mathbb{R} \to \mathbb{R}^+ \) be the investor’s initial desired allocation in the risky asset and let the function \( r_0 : \mathbb{R} \to \mathbb{R}^+ \) be given by

\[
r_0(x) = \frac{\sigma_0}{\lambda_0} \pi^*_0(x). \tag{33}
\]

Let us assume that the autonomous equation

\[
r_0(h_0(x)) = h_0'(x) \tag{34}
\]
has a solution $h_0 : \mathbb{R} \to \mathbb{R}$ of the form

$$h_0(x) = \int_{\mathbb{R}} \frac{e^{yx} - 1}{y} \nu(dy)$$  \hspace{1cm} (35)$$

where $\nu$ satisfies (32) and is such that $\text{Range}(h_0) = (-\infty, +\infty)$. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx} - \frac{1}{2} y^2}{y} \nu(dy)$$  \hspace{1cm} (36)$$

and define the processes $X^*_t$ and $\pi^*_t$, $t \geq 0$, by

$$X^*_t = h\left(h^{[-1]}(x,0) + A_t + M_t\right)$$  \hspace{1cm} (37)$$

and

$$\pi^*_t = \frac{\lambda_t}{\sigma_t} h_x \left(h^{[-1]}(X^*_t, A_t)\right),$$  \hspace{1cm} (38)$$

with $A_t$ and $M_t$, $t \geq 0$, as in (9) and (31), respectively. Consider the function $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by

$$r(x,t) = h_x \left(h^{[-1]}(x,0), t\right)$$  \hspace{1cm} (39)$$

and the process

$$R^*_t = r(X^*_t, A_t).$$  \hspace{1cm} (40)$$

Finally, define $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{[-1]}(x,s)} + r(x,s) ds + \int_0^x e^{-h^{[-1]}(z,0)} dz$$  \hspace{1cm} (41)$$

where $r$ and $h$ are as in (39) and (36), respectively. Then,

(i) the portfolio $\pi^*_t$ generates the wealth process $X^*_t$,

(ii) the processes $X^*_t$ and $R^*_t$ solve the autonomous system (13),

(iii) the policy $\pi^*_t$, $t \geq 0$, is the optimal portfolio process for the forward investment performance criterion

$$U(x,t) = u(x, A_t)$$  \hspace{1cm} (42)$$

with $u(x,t)$ as in (41).

**Proof.** i) Let $N_t = h^{[-1]}(x,0) + A_t + M_t$. Applying Itô’s formula to $X^*_t$ yields

$$dX^*_t = h_x (N_t, A_t) dN_t + \lambda_t^2 \left(h_t (N_t, A_t) + \frac{1}{2} h_{xx} (N_t, A_t)\right) dt$$

$$= h_x (N_t, A_t) \lambda_t (\lambda_t dt + dW_t) + \lambda_t^2 \left(h_t (N_t, A_t) + \frac{1}{2} h_{xx} (N_t, A_t)\right) dt.$$
From Proposition 7 we recall that $h$ solves the heat equation (15). Therefore, the above simplifies to

$$dX_t^* = h_x (N_t, A_t) \lambda_t (\lambda_t \, dt + dW_t).$$

On the other hand,

$$h_x (N_t, A_t) = h_x \left( h^{(-1)} \left( h^{(-1)} (x, 0) + A_t + M_t, A_t \right), A_t \right)$$

$$= h_x \left( h^{(-1)} (X^*_t, A_t), A_t \right)$$

and using (3) and (38) we easily conclude.

ii) The above calculations yield that $X_t^*$ satisfies the first equation in (13). For the second equation we have

$$dR_t^* = dr (X_t^*, A_t)$$

$$= \lambda_t^2 \left( r_t (X_t^*, A_t) + \frac{1}{2} r^2 (X_t^*, A_t) r_{xx} (X_t^*, A_t) \right) \, dt + r_x (X_t^*, A_t) dX_t^*.$$  

Proposition 4 and (39) yield that the function $r (x, t)$ solves the fast diffusion equation (11). Thus, the above drift vanishes and the assertion follows.

iii) This part has several assertions to be established. The arguments are lengthy and the calculations rather tedious. Moreover, most of them follow from arguments used in the proof of Theorem 4 in [6]. For these reasons, we only highlight the main steps.

To this end, we first need to show that the policy in consideration $\pi_t^*, t \geq 0$, defined in (38) is admissible. Its $\mathcal{F}_t$–measurability follows trivially. To show the integrability property $E \left( \int_0^t |\sigma_s \pi_s|^2 \, ds \right) < \infty$, $t > 0$, we use (20) and that

$$h_x (x, t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2} y^2 t} \nu (dy),$$

as it follows by direct differentiation and Tonelli’s theorem. Writing

$$\left( h_x \left( h^{(-1)} (X_t^*, A_t), A_t \right) \right)^2$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(y_1 + y_2) (h^{(-1)} (x, 0) + A_t + M_t) - \frac{1}{2} (y_1^2 + y_2^2) A_t \nu (dy_1) \nu (dy_2),}$$

using (38), (32) and the uniform bound on the market risk premium we conclude.

To show that $U (x, t)$ is a forward investment performance process, it suffices to check that $u$ given in (41) solves (8) and that $E \left( U (X_t^*, t)^+ \right) < +\infty$. Finally, one needs to show that the policy is optimal. For this, one needs to prove that the drift of the process $u (X_t^*, A_t)$ vanishes.

Next, we state the results for the semi-infinite case. Due to space limitations, we compress some arguments.
Theorem 10 Let $\pi_0^* : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be his initial desired allocation in the risky asset. Define the function $r_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as in $(34)$ and assume that equation $(35)$ has a solution $h_0 : \mathbb{R} \rightarrow \mathbb{R}^+$ of the form
\[ h_0(x) = \int_{0^+}^{+\infty} e^{yx} \nu(dy) \] (44)
with $\nu \in B_+^1(\mathbb{R})$ with $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ and satisfying $(32)$. Let $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}^+$ be given by
\[ h(x, t) = \int_{0^+}^{+\infty} e^{yx - \frac{1}{2}y^2t} \nu(dy) . \] (45)
Then the processes $X_t^* \text{ and } \pi_t^*$, $t \geq 0$, given by
\[ X_t^* = h \left( h^{(-1)} (x, 0) + A_t + M_t, A_t \right) \quad \text{and} \quad \pi_t^* = \frac{\lambda_t}{\sigma_t} h_x \left( h^{(-1)} (X_t^*, A_t), A_t \right) \]
are optimal. Moreover, $X_t^* > 0$, $t \geq 0$, a.e. Let $r : \mathbb{R}^+ \times [0, +\infty) \rightarrow \mathbb{R}^+$ be given by $r(x, t) = h_x \left( h^{(-1)} (x, t), t \right)$.
If $\nu ((0, 1]) = 0$ and $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y - t} < +\infty$, define $u : \mathbb{R}^+ \times [0, +\infty) \rightarrow \mathbb{R}$ by
\[ u(x, t) = \frac{1}{2} \int_{0}^{t} e^{-h^{(-1)}(x, s) + \frac{1}{2}r(x, s)} ds + \int_{0}^{x} e^{-h_0^{(-1)}(z)} dz, \] (46)
while if $\nu ((0, 1]) > 0$ and/or $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y - t} = +\infty$, define
\[ u(x, t) = \frac{1}{2} \int_{0}^{t} e^{-h^{(-1)}(x, s) + \frac{1}{2}r(x, s)} ds + \int_{x_0}^{x} e^{-h_0^{(-1)}(z)} dz, \] (47)
for $x_0 > 0$. Then, $X_t^*$ and $\pi_t^*$ are optimal with respect to the forward investment criterion $U(x, t) = u(x, A_t)$ with $u(x, t)$ as above.

The above theorems give answers to the questions we put forward in section 2. Indeed, the desired initial allocation $\pi_0^*(x)$ yields the initial risk tolerance $r_0(x)$. If equation $(34)$ has a solution that can be represented in the appropriate integral form, then the measure $\nu$ is identified. From there, one readily constructs the harmonic function $h(x, t)$.

Using $h(x, t)$, $h_x(x, t)$, the market input processes $A_t$ and $M_t$, and formulae $(37)$ and $(38)$ we find the processes $X_t^*$ and $\pi_t^*$, $t > 0$. From $h(x, t)$, $h_x(x, t)$, $(39)$ and (41), one, also, finds the differential input $u(x, t)$ and, in turn, the time-monotone investment performance criterion from (42). The theorems yield that the processes $X_t^*$ and $\pi_t^*$ are optimal with respect to the forward investment process $U(x, t) = u(x, A_t)$.

Conditions $(35)$ and $(44)$ yield the admissibility requirement for the desired initial allocations.
5.1 Inferring risk preferences from desired asset allocations

The analysis herein highlights how risk preferences can be entirely inferred from the investor’s allocations. This is in contrast to the traditional approach in portfolio theory in which one first specifies the utility and then determines the optimal policies. Herein, one takes the reverse route.

Indeed, the investment criterion is found by (42) as long as the market risk premium is specified (cf. (9)) and the function \( u(x,t) \) is known. The latter is given by (41) (resp. (46) and (47)) which requires knowledge of \( h(x,t) \) and \( r(x,t) \). Both functions are specified once the measure \( \nu \) is known, as it is manifested in (36) (resp. (45)) and (39). The measure is found by (34) and (35).

It is worth mentioning an alternative way which provides a direct construction of \( u(x,t) \) from \( r(x,t) \) which entirely bypasses the use of \( h(x,t) \), \( t > 0 \). This result carries useful insights, for it essentially shows how risk preferences can be recovered simply by propagating the initial risk preferences along the inverted characteristic curves whose slope is half of the local risk tolerance. On the other hand, the slope of these curves is directly related to the optimal portfolio itself as it can be seen from (38). In other words, the construction below alludes to the fact that the investment performance process \( U(x,t) \) can be directly constructed from the optimal portfolio process \( \pi^*_t \) by inverting the appropriate stochastic characteristic curves. This idea was recently explored by El Karoui.

The arguments that follow are non-formal and are presented only to highlight the approach. The main idea comes from the fact that the fully nonlinear equation (8) can be equivalently written as the one below if one takes into account (30). Similarly, the initial condition in (48) comes from (29).

To this end, let us assume that the function \( r(x,t) \) is known for \( t \geq 0 \) and consider the equation

\[
\frac{du}{dt} + \frac{1}{2} r(x,t) u_x = 0 \quad \text{with} \quad u(x,0) = \int_0^x e^{-h_0^{(-1)}(z)} dz, \quad (48)
\]

with \( h_0(x) \) as in (35).

Classical results in the theory of first order linear partial differential equations yield that

\[
u(x,t) = u_0(X^{(-1),x}(t))
\]

where the \( X^{(-1)}(x,t) \) denotes the inverse of the characteristic curve of (48) which starts at \( t = 0 \) at point \( x \),

\[
\frac{dX^x(t)}{dt} = \frac{1}{2} r(X^x(t), t) \quad \text{with} \quad X^x(0) = x. \quad (49)
\]

We provide an example of this construction in the first example below.
6 Examples

In this section we provide some representative examples for both the unconstrained and constrained wealth cases. For the sake of simplicity we present the results in terms of the initial risk tolerance and not the initial allocation in (33). This is only to eliminate the multiplicative constant $\frac{\lambda_t}{\sigma_t}$.

6.1 \textit{Domain} ($r = (-\infty, +\infty)$)

- \( r_0 (x) = 1 \)

Equation (34) has the solution \( h_0 (x) = x \). From (35) we easily deduce that \( \nu = \delta_0 \), where \( \delta_0 \) is a Dirac measure at 0. Then (36) yields \( h(x,t) = x \). In turn, (37) and (38) give

\[
X^*_t = x + A_t + M_t \quad \text{and} \quad \pi^*_t = \frac{\lambda_t}{\sigma_t}.
\]

On the other hand, (39) implies \( r(x,t) = 1 \). From (41) we deduce

\[
u = 0; \quad \text{where} \quad \delta_0 \text{ is a Dirac measure at } 0.
\]

Then (36) yields \( h(x,t) = x \).

In turn, (37) and (38) give

\[
X^*_t = x + \frac{1}{2} t.
\]

This, in turn, yields \( h(x,t) = x \).

Therefore, the solution takes, along the characteristics, the form

\[
u = 0; \quad \text{where} \quad \delta_0 \text{ is a Dirac measure at } 0.
\]

The optimal processes in (37) and (38) turn out to be

\[
X^*_t = b e^{-\frac{1}{2} a^2 t} \left( \sinh \left( \frac{a}{b} x + \sqrt{\frac{a^2}{b^2} x^2 e^{a^2 t} + 1 + a (A_t + M_t)} \right) \right)\]
and

\[ \pi^*_t = \frac{\lambda_t}{\sigma_t} \sqrt{a^2 (X_t^*)^2 + b^2 e^{-a^2 A_t}}. \]

If, \( a = 1 \), (41) gives

\[ u(x,t) = \frac{1}{2} \left( \ln \left( x + \sqrt{x^2 + b^2 e^{-t}} \right) - \frac{e^t}{b^2} \frac{x - \sqrt{x^2 + b^2 e^{-t}} - t}{2} \right) - \frac{1}{2} \ln b, \]

while, if \( a \neq 1 \),

\[ u(x,t) = \frac{\sqrt{a}}{a^2 - 1} e^{\frac{a}{2} \frac{b^2 e^{-at} + a(1 + a) (ax^2 + x \sqrt{a^2 x^2 + b^2 e^{-at}})}{a^2 x^2 + b^2 e^{-at}}} - \frac{\sqrt{a}}{a^2 - 1} b^{1 - \frac{a}{2}}. \]

Further results on this case can be found in [15]. This two-parameter class is quite rich and leads to the popular utilities - power, logarithmic and exponential - for limiting values of the parameters \( a \) and \( b \) (see remark 12 at the end of this section).

• \( r_0(x) = f\left(F^{(-1)}(x)\right) \) with \( F(x) = \int_0^x e^{\frac{1}{2} z^2} dz, \ x \in \mathbb{R} \) and \( f = F' \).

Equation (34) has the solution \( h_0(x) = F(x) \) which corresponds to the measure \( \nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy. \) Then, \( h(x,t) = F\left(\frac{x - t}{\sqrt{t+1}} \right). \) Using that \( h^{(-1)}(x,t) = \sqrt{t+1} F^{(-1)}(x) \) we obtain \( h_x \left(h^{(-1)}(x,t), t\right) = \frac{1}{\sqrt{t+1}} f\left(F^{(-1)}(x)\right). \)

The optimal processes are given by

\[ X^*_t = F\left(\frac{F^{(-1)}(x) + A_t + M_t}{\sqrt{A_t + 1}}\right) \]

and

\[ \pi^*_t = \frac{1}{\sqrt{A_t + 1}} f\left(\frac{F^{(-1)}(x) + A_t + M_t}{\sqrt{A_t + 1}}\right). \]

We deduce (after tedious calculations that are omitted) that

\[ u(x,t) = k_1 F\left(F^{(-1)}(x) - \sqrt{t+1}\right) + k_2 \]

with \( k_1 = e^{-\frac{1}{2}} \) and \( k_2 = e^{-\frac{1}{2}} \int_0^x e^{\frac{1}{2} z^2} dz. \)

6.2 Domain \((r(x)) = (0, +\infty) \) or \([0, +\infty)\)

• \( r_0(x) = \gamma x \), \( \gamma > 1 \)

Equation (34) yields \( h_0(x) = \frac{1}{\gamma} e^{\gamma x} \). From (35) we then have \( \nu = \delta_\gamma. \) Therefore \( h(x,t) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2} \gamma^2 t}, h^{(-1)}(x,t) = \ln (\gamma x)^{\frac{1}{\gamma} + \frac{1}{2} \gamma t} \) and \( h_x \left(h^{(-1)}(x,t), t\right) = \gamma x. \) In turn, (37) and (38) yield

\[ X^*_t = x \exp \left(\left(\gamma - \frac{1}{2} \gamma^2\right) A_t + \gamma M_t\right) \quad \text{and} \quad \pi^*_t = \frac{\lambda_t}{\sigma_t} X^*_t. \]
Since \( \nu ((0,1]) = 0 \), \( u \) is given by (24),
\[
u(x,t) = -\frac{1}{2} \int_0^t \gamma x e^{-\left(\ln(x)\frac{1}{2} + \frac{1}{2}\gamma s\right)} + \frac{s}{x} ds + \int_0^x (\gamma s) - \frac{1}{\gamma - 1} ds = \frac{\gamma - 1}{\gamma - 1} x^{\frac{1}{2} - \frac{1}{\gamma}} e^{-\frac{1}{2} x}.
\]

- \( r_0(x) = \gamma x, \gamma \in (0,1) \)

This example is the same as the above. The only difference is that now \( \nu ((0,1]) \neq 0 \). Thus, \( u \) is given by (47) for for \( x_0 > 0 \),
\[
u(x,t) = -\frac{1}{2} \int_0^t \gamma x e^{-\left(\ln(x)\frac{1}{2} + \frac{1}{2}\gamma s\right)} + \frac{s}{x} ds + \int_0^x (\gamma s) - \frac{1}{\gamma - 1} ds = -\frac{\gamma - 1}{\gamma - 1} x^{\frac{1}{2} - \frac{1}{\gamma}} e^{-\frac{1}{2} x}.
\]

- \( r_0(x) = 1 \)

Equation (34) yields \( h_0(x) = e^x \) which corresponds to \( \nu = \delta_1 \). Then, \( h(x,t) = e^{x - \frac{1}{2} t}, h^{(-1)}(x,t) = \ln x + \frac{1}{2} t \), and \( h_x(h^{(-1)}(x,t),t) = x \). In turn,
\[
X_t^* = x \exp \left( \frac{1}{2} \Delta_t + M_t \right) \quad \text{and} \quad \pi_t^* = \frac{\lambda_t}{\sigma_t} X_t^*.
\]

Since \( \nu ((0,1]) \neq 0 \), \( u \) is given by (26) for \( x_0 > 0 \),
\[
u(x,t) = -\frac{1}{2} \int_0^t x e^{-(\ln(x)\frac{1}{2} + \frac{1}{2}) + \frac{s}{x}} ds + \int_{x_0}^x \frac{1}{z} dz = \ln \frac{x}{x_0} - \frac{t}{2}.
\]

**Remark 11** The examples in 6.2 are the ones examined in an infinitesimal setting in [2].

**Remark 12** It is worth noticing that the examples in 6.2 can be thought as limiting cases of the parameters \( a \) and \( b \) in \( r_0(x) = \sqrt{ax^2 + b} \). Indeed, the first two correspond to \( b = 0 \) and \( \gamma = \sqrt{a} \) while the last one to \( b = 0 \) and \( \gamma = 1 \). The first example in 6.1 can be also thought as a limiting case, namely, for \( a = 0 \) and \( b = 1 \).

**References**


