SPDE and portfolio choice

(joint work with M. Musiela)

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Performance measurement
Market environment

Riskless and risky securities

- \((\Omega, \mathcal{F}, \mathbb{P})\); \(W = (W^1, \ldots, W^d)\) standard Brownian Motion

- Traded securities

  \[
  1 \leq i \leq k \quad \left\{ \begin{aligned}
  dS^i_t &= S^i_t(\mu^i_t dt + \sigma^i_t \cdot dW^i_t) , \\
  dB^i_t &= r^i_t B^i_t dt ,
  \end{aligned} \right. \quad S^i_0 > 0 \quad \text{bounded and } \mathcal{F}_t\text{-measurable stochastic processes}
  \]

- Postulate existence of an \(\mathcal{F}_t\)-measurable stochastic process \(\lambda_t \in \mathbb{R}^d\) satisfying

  \[
  \mu_t - r_t \mathbf{1} = \sigma^T_t \lambda_t
  \]

- No assumptions on market completeness
Market environment

- Self-financing investment strategies $\pi_t^0, \pi_t^i, \ i = 1, \ldots, k$

- Present value of this allocation

\[ X_t = \sum_{i=0}^{k} \pi_t^i \]

\[ dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t \, dt + dW_t) \]

\[ = \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t) \]

\[ \pi_t = (\pi_t^1, \ldots, \pi_t^k) \]
Investment performance process

$U_t(x)$ is an $\mathcal{F}_t$-adapted process, $t \geq 0$

- The mapping $x \rightarrow U_t(x)$ is increasing and concave

- For each self-financing strategy, represented by $\pi$, the associated (discounted) wealth $X_t$ satisfies

  $$E_P(U_t(X_t^\pi) \mid \mathcal{F}_s) \leq U_s(X_s^\pi), \quad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by $\pi^*$, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

  $$E_P(U_t(X_t^{\pi^*}) \mid \mathcal{F}_s) = U_s(X_s^{\pi^*}), \quad 0 \leq s \leq t$$
Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

Backwards in time generation of optimal performance

$$V_t(x) = \sup_\pi E_\mathbb{P}(u_T(X_T^\pi)|\mathcal{F}_t; X_t^\pi = x)$$

$$V_t(x) = \sup_\pi E_\mathbb{P}(V_s(X_s^\pi)|\mathcal{F}_t; X_t^\pi = x) \quad \text{(DPP)}$$

$$V_t(x) = E_\mathbb{P}(V_s(X_s^{\pi^*})|\mathcal{F}_t; X_t^{\pi^*} = x)$$

$$\Downarrow$$

$$U_t(x) \equiv V_t(x) \quad 0 \leq t < T$$

The performance process coincides with the traditional value function
Alternative framework

A datum $u_0(x)$ is assigned at the beginning of the trading horizon, $t = 0$

$$U_0(x) = u_0(x)$$

Forward in time generation of optimal performance

$$E_P(U_t(X_t^\pi)|\mathcal{F}_s) \leq U_s(X_s^\pi), \quad 0 \leq s \leq t$$

$$E_P(U_t(X_t^{\pi^*})|\mathcal{F}_s) = U_s(X_s^{\pi^*}), \quad 0 \leq s \leq t$$
Desired properties of performance process

- Defined at all times, not tied down to a specific horizon $T$
- Direct implementation of a benchmark and alternative market views
- Associated portfolios have an intuitive and universal structure

Not exogenously given. It follows the market movements and investment opportunities “path by path”

Many difficulties due to “inverse in time” nature of the problem
The stochastic PDE of the forward performance process
Intuition for the structure of the forward performance process

• Assume that $U = U(x, t)$ solves

$$dU(x, t) = b(x, t) \, dt + a(x, t) \cdot dW_t$$

where $b, a$ are $\mathcal{F}_t$—measurable processes.

• Recall that for an arbitrary admissible portfolio $\pi$, the associated wealth process, $X^{\pi}$, solves

$$dX^{\pi}_t = \sigma_t \pi_t (\lambda_t dt + dW_t)$$

• Apply the Ito-Ventzell formula to $U(X^{\pi}_t, t)$ we obtain

$$dU(X^{\pi}_t, t) = b(X^{\pi}_t, t) \, dt + a(X^{\pi}_t, t) \cdot dW_t$$

$$+ U_x(X^{\pi}_t, t) dX^{\pi}_t + \frac{1}{2} U_{xx}(X^{\pi}_t, t) d\langle X^{\pi} \rangle_t + a_x(X^{\pi}_t, t) d\langle W, X^{\pi} \rangle_t$$

$$= \left( b(X^{\pi}_t, t) + U_x(X^{\pi}_t, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x(X^{\pi}_t, t) + \frac{1}{2} U_{xx}(X^{\pi}_t, t) |\sigma_t \pi_t|^2 \right) dt$$

$$+ (a(X^{\pi}_t, t) + U_x(X^{\pi}_t, t) \sigma_t \pi_t) \cdot dW_t$$
Intuition (continued)

• By the monotonicity and concavity assumptions, the quantity

\[ \sup_{\pi} \left( U_x (X_{t}^{\pi}, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_{t}^{\pi}, t) + \frac{1}{2} U_{xx} (X_{t}^{\pi}, t) |\sigma_t \pi_t|^2 \right) \]

is well defined.

• Calculating the optimum \( \pi^* \) yields

\[ \pi_t^* = -\sigma_t^+ \left( \frac{U_x (X_{t}^{\pi^*}, t) \lambda_t + a_x (X_{t}^{\pi^*}, t)}{U_{xx} (X_{t}^{\pi^*}, t)} \right) \]

• Deduce that the above supremum is given by

\[ M^* (X_{t}^{\pi^*}, t) = -\frac{1}{2} \left| \frac{\sigma_t \sigma_t^+ (U_x (X_{t}^{\pi^*}, t) \lambda_t + a_x (X_{t}^{\pi^*}, t))}{U_{xx} (X_{t}^{\pi^*}, t)} \right|^2 \]

• Choose the drift coefficient

\[ b (x, t) = -M^* (x, t) \]
The forward performance SPDE

Let $U(x, t)$ be an $\mathcal{F}_t$–measurable process such that the mapping $x \to U(x, t)$ is increasing and concave. Let also $U = U(x, t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2} \left| \sigma \sigma^+ \mathcal{A} (U \lambda + a) \right|^2 + a \cdot dW$$

where $a = a(x, t)$ is an $\mathcal{F}_t$–measurable process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x, t)$ is a forward performance process.

The process $a$ may depend on $t, x, U$, its spatial derivatives etc.
At the optimum

- The optimal portfolio vector $\pi^*$ is given in the feedback form

$$\pi_t^* = \pi^*(X^*_t, t) = -\sigma + \frac{A(U \lambda + a)}{A^2 U}(X^*_t, t)$$

- The optimal wealth process $X^*$ solves

$$dX^*_t = -\sigma \sigma + \frac{A(U \lambda + a)}{A^2 U}(X^*_t, t)(\lambda dt + dW_t)$$
Solutions to the forward performance SPDE

\[ dU = \frac{1}{2} \left| \sigma \sigma^+ A (U \lambda + a) \right|^2 + a \cdot dW \]

Local differential coefficients

\[ a (x, t) = F (x, t, U (x, t), U_x (x, t)) \]

**Difficulties**

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on \( U_x \)
- The equation is not (degenerate) elliptic
Choices of volatility coefficient

• \( a(x, t) = 0 \)

The forward performance SPDE simplifies to

\[
dU = \frac{1}{2} \frac{\left| \sigma \sigma^+ A(U \lambda) \right|^2}{A^2 U} dt
\]

The process

\[
U = U(x, t) = u(x, A_t) \quad \text{with} \quad A_t = \int_0^t \left| \sigma_s \sigma^+_s \lambda_s \right|^2 ds
\]

with \( u : \mathcal{R} \times [0, +\infty) \rightarrow \mathcal{R} \), increasing and concave with respect to \( x \), and solving

\[
 u_t u_{xx} = \frac{1}{2} u_x^2
\]

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)
\( a(x, t) = 0 \)

\( \sigma, \lambda \) constants and \( u \) separable (in space and time)

The forward performance process reduces to a deterministic function.

\[
U = U(x, t) = u(x, t)
\]

\[
u(x, t) = -e^{-x + \frac{t}{2}} \quad \text{or} \quad u(x, t) = \frac{1}{x} e^{-\frac{\gamma}{2(1-\gamma)} \lambda^2 t}
\]

Horizon-unbiased utilities

Henderson-Hobson (2006)
The “market-view” case

\[ \alpha = U\phi, \quad \phi \text{ is a } d\text{-dim } \mathcal{F}_t\text{-mble process} \]

- The forward performance SPDE becomes

\[
dU = \frac{1}{2} \frac{\left| \sigma \sigma^+ A U (\lambda + \phi) \right|^2}{A^2 U} dt + U \phi \cdot dW
\]

- Define the processes \( Z \) and \( A \) by

\[
dZ = Z\phi \cdot dW \quad \text{and} \quad Z_0 = 1
\]

and

\[
A_t = \int_0^t \left| \sigma_s \sigma^+_s (\lambda_s + \phi_s) \right|^2 ds
\]

- The process \( U = U(x, t) \)

\[
U(x, t) = u(x, A_t) Z_t
\]

with \( u \) solving

\[
u_t u_{xx} = \frac{1}{2} u_x^2
\]

is a solution
The “benchmark” case

\[ \alpha(x, t) = -x U(x, t) \delta, \quad \delta \text{ is a } d\text{--dim } \mathcal{F}_t\text{--mble process} \]

- The forward performance SPDE becomes

\[
dU(x, t) = \frac{1}{2} \left| \sigma_t \sigma^+_t (U_x(x, t) (\lambda_t - \delta_t) - x U_{xx}(x, t)) \right|^2 U_{xx}(x, t) \frac{dt}{U_{xx}(x, t)} - x U_x(x, t) \delta_t \cdot dW_t
\]

- Define the processes \( Y \) and \( A \) by

\[
dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1
\]

and

\[
A_t = \int_0^t \left| \sigma_s \sigma^+_s \lambda_s - \delta_s \right|^2 ds.
\]

- Assume \( \sigma \sigma^+ \delta = \delta \)

- The process

\[
U = U(x, t) = u \left( \frac{x}{Y_t}, A_t \right)
\]

with \( u \) as before is a forward performance.
A general case

\[ \alpha(x, t) = -x U_x(x, t) \delta + U(x, t) \phi \]

- The forward performance SPDE becomes

\[
dU(x, t) = \frac{1}{2} \left[ \sigma_t \sigma_t^+ (U_x(x, t) ((\lambda_t + \phi_t) - \delta_t) - x U_{xx}(x, t) \delta_t) \right]^2 dt \\
+ (-x U_x(x, t) \delta_t + U(x, t) \phi_t) \cdot dW_t
\]

- Recall the "benchmark" and "market view processes"

\[
dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y = 1
\]

and

\[
dZ_t = Z_t \phi_t \cdot dW_t \quad \text{with} \quad Z = 1
\]
Define the process

\[ A_t = \int_0^t \left| \sigma_s \sigma^+_s (\lambda_s + \phi_s) - \delta_s \right|^2 ds \]

The process

\[ U = U(x, t) = u \left( \frac{x}{Y_t}, A_t \right) Z_t \]

is a forward performance

MZ (2006, 2007)
The u-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \quad t > 0,$$

with $u_0 (x) = U (x, 0)$.

The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = - \frac{u_x}{u_{xx}}$$

Modelling considerations
Three related pdes

- Fast diffusion equation for risk tolerance

\[
\begin{aligned}
rt + \frac{1}{2}r^2r_{xx} &= 0 \\
\Gamma(x, 0) &= \Gamma_0(x)
\end{aligned}
\]  

Conductivity : \( r^2 \)

- The transport equation

\[
\begin{aligned}
Ut + \frac{1}{2}ru_x &= 0 \\
\text{with } u_0 \text{ such that } \Gamma_0 = r(x, 0) = -\frac{u_0'(x)}{u_0''(x)}
\end{aligned}
\]

- Porous medium equation for risk aversion  \( \gamma = r^{-1} \)

\[
\begin{aligned}
\gamma_t &= \frac{1}{2}F(\gamma)_{xx} \\
\text{with } F(\gamma) &= \gamma^{-1}
\end{aligned}
\]
Difficulties

- **Differential input equation:** \( u_t \ u_{xx} = \frac{1}{2} u_x^2 \)
  
  Inverse problem and fully nonlinear

- **Transport equation:** \( u_t + \frac{1}{2} r u_x = 0 \)
  
  Shocks, solutions past singularities

- **Fast diffusion equation:** \( r_t + \frac{1}{2} r^2 r_{xx} = 0 \)
  
  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- **Porous medium equation:** \( \gamma_t = \frac{1}{2} (\frac{1}{\gamma})_{xx} \)
  
  Majority of results for (PME), \( \gamma_t = (\gamma^m)_{xx} \), are for \( m > 1 \), partial results for \(-1 < m < 0\)
An example of local risk tolerance
(MZ (2006) and Z-Zhou (2007))

\[ r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0 \]

(Very) special cases

\[ r(x, t; 0, \beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in R \]
\[ r(x, t; 1, 0) = |x| \quad \longrightarrow \quad u(x, t) = \log x - \frac{t}{2}, \quad x > 0 \]
\[ r(x, t; \alpha, 0) = \sqrt{\alpha} |x| \quad \longrightarrow \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)}t}, \quad x \geq 0, \gamma = \frac{\sqrt{\alpha-1}}{\sqrt{\alpha}} \]
Optimal allocations
Optimal portfolio vector

- The SPDE for the forward performance process

\[ dU = \frac{1}{2} \left| \sigma \sigma^+ A(U\lambda + a) \right|^2 dt + a \cdot dW \]

- The optimal portfolio vector

\[ \pi_t^* = \pi^*(t, X_t^*) = -\sigma^+ \frac{A(U\lambda + a)}{A^2U} (X_t^*, t) \]

- The optimal wealth process

\[ dX_t^* = -\sigma \sigma^+ \frac{A(U\lambda + a)}{A^2U} (X_t^*, t) (\lambda dt + dW_t) \]
Optimal portfolios in the MZ example
Forward performance process

Stochastic market input

\[ \lambda_t, \sigma_t \]

\[ \downarrow \]

benchmark, views

\[ (Y_t, Z_t, A_t) \]

Differential input

\[ x, r_0(x) \text{ or } u_0(x) \]

\[ \downarrow \]

\[ r_t + \frac{1}{2} r^2 r_{xx} = 0 \] (FDE)

\[ u_t + \frac{1}{2} ru_x = 0 \] (TE)

\[ u(x, t) \]

Solution to SPDE

\[ U(x, t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t \]
The structure of optimal portfolios

\[ dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t) \]

**Stochastic input**

**Market**

\( (Y_t, Z_t, A_t) \)
\( \lambda_t, \sigma_t, \delta_t, \phi_t \)

**Differential input**

**Individual**

wealth \( x \)
risk tolerance \( r(x, t) \)

\[ \frac{1}{Y_t} \pi_t^* \] is a *linear combination*

of (benchmarked) optimal wealth

and subordinated (benchmarked) risk tolerance
**Optimal asset allocation**

- Let $X^*_t$ be the optimal wealth, $Y_t$ the benchmark and $A_t$ the time-rescaling processes

\[
\begin{align*}
    dX^*_t &= \sigma_t \pi^*_t \cdot (\lambda_t dt + dW_t) \\
    dY_t &= Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\
    dA_t &= |\sigma_t \sigma^+_t (\lambda_t + \phi_t) - \delta_t|^2 dt
\end{align*}
\]

- Define

\[
\begin{align*}
    \tilde{X}^*_t &\triangleq \frac{X^*_t}{Y_t} \\
    \tilde{R}^*_t &\triangleq r(\tilde{X}^*_t, A_t)
\end{align*}
\]

**Optimal (benchmarked) portfolios**

\[
\hat{\pi}^*_t \triangleq \frac{1}{Y_t} \pi^*_t = m_t \tilde{X}^*_t + n_t \tilde{R}^*_t
\]

\[
\begin{align*}
    m_t &= \sigma^+_t \delta_t \\
    n_t &= \sigma^+_t (\lambda_t + \phi_t - \delta_t)
\end{align*}
\]
Decomposition of optimal investment strategies

\[ \tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* + \sigma_t^+ (\lambda_t + \phi_t - \delta_t) \tilde{R}_t^* \]

- No benchmark: \( \delta \equiv 0 \)
  \[ \pi_t^* = \sigma_t^+ (\lambda_t + \phi_t) R_t^* \]

- Tracking the benchmark: \( \lambda + \phi - \delta = 0 \)
  \[ \tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* \quad X_t^* = xY_t \]

- No benchmark and market neutralization: \( \delta = 0 \) and \( \lambda + \phi = 0 \)
  \[ \pi_t^* = 0 \quad X_t^* = x \]

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Stochastic evolution of wealth-risk tolerance
A system of SDEs at the optimum

\[ \tilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \tilde{R}_t^* = r(\tilde{X}_t^*, A_t) \]

\[
\begin{cases}
  d\tilde{X}_t^* = \tilde{R}_t^*(\sigma_t\sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\
  d\tilde{R}_t^* = r_x(\tilde{X}_t^*, A_t) d\tilde{X}_t^*
\end{cases}
\]

- Separability of wealth dynamics in terms of risk tolerance and market input
- Sensitivity of risk tolerance in terms of its spatial gradient and changes in optimal wealth
An example
Asymptotically linear local risk tolerance

(Z.-Zhou (2007))

Recall that the local risk tolerance function $r$ solves the fast diffusion equation

$$rt + \frac{1}{2}r^2r_{xx} = 0$$

A two-parameter family

$$r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \quad x \in \mathcal{R}, \ t > 0 \quad \text{and} \quad \alpha, \beta > 0.$$ 

- $r(x, t; 0, \beta) \rightarrow$ exponential utility
- $r(x, t; \alpha, 0), \ \alpha \neq 1 \rightarrow$ power utility for $x \in [0, \infty)$
- $r(x, t; 1, 0) \rightarrow$ logarithmic utility for $x \in (0, \infty)$
Forward performance process

\[ U(x, t) = u \left( \frac{x}{Y_t}, A_t \right) Z_t \]

where

\[ -\frac{u_x(x, t)}{u_{xx}(x, t)} = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \quad x \in \mathcal{R}, \ t > 0. \]

We may also use the transport equation

\[ u_t + \frac{1}{2} r(x, t; \alpha, \beta) u_x = 0. \]

The function \( u \) is globally defined as opposed to its traditional power and logarithmic counterparts.
System of optimal wealth and risk tolerance

- The (benchmarked) processes \( \tilde{X}^* = \frac{X}{Y} \) and \( \tilde{R}^* = r(\tilde{X}^*, A) \) solve

\[
d\tilde{X}_t^* = \tilde{R}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t) dt + dW_t)
\]

\[
d\tilde{R}_t^* = \alpha \tilde{X}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t) dt + dW_t)
\]

with \( \tilde{X}_0^* = x \) and \( \tilde{R}_0^* = r(x, 0) = \sqrt{\alpha x^2 + \beta} \) and \( n = \sigma^+ (\lambda + \phi - \delta) \).

- In turn, for \( t > 0 \),

\[
\tilde{X}_t^* = e^{-\frac{\alpha}{2} \int_0^t |\sigma_s n_s|^2 ds} \left( x \cosh (\sqrt{\alpha k_t}) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh (\sqrt{\alpha k_t}) \right)
\]

and

\[
\tilde{R}_t^* = e^{-\frac{\alpha}{2} \int_0^t |\sigma_s n_s|^2 ds} \left( \sqrt{\alpha} x \sinh (\sqrt{\alpha k_t}) + \sqrt{\alpha x^2 + \beta} \cosh (\sqrt{\alpha k_t}) \right),
\]

where

\[
k_t = \int_0^t \sigma_s \sigma_s^+ (\lambda_s + \phi_s - \delta_s) \cdot ((\lambda_s - \delta_s) ds + dW_s).
\]
Vector of optimal asset allocations

\[ \tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* + \sigma_t^+ (\lambda_t + \phi_t - \delta_t) \tilde{R}_t^* \]

Special cases

No benchmark: \( \delta = 0 \)

\[ \tilde{\pi}_t^* = \sigma_t^+ (\lambda_t + \phi_t) \tilde{R}_t^* \]

• Riskless optimal allocation: \( \lambda + \phi = 0 \)
  \[ \pi_t^* = 0 \quad \text{and} \quad \pi_{t,0}^* = x \]

• No riskless component in optimal allocation: \( 1 \cdot \sigma^+ (\lambda + \phi) = 1 \)
  \[ \pi_{t,0}^* = 0 \quad \text{and} \quad 1 \cdot \pi_t^* = X_t^* \]

• Arbitrary allocations: \( 1 \cdot \sigma^+ (\lambda + \phi) = p \)
  \[ \pi_{t,0}^* = (1 - p) X_t^* \quad \text{and} \quad 1 \cdot \pi_t^* = p X_t^* \]
References

(joint with M. Musiela)

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