

DETERMINING FUCHSIAN GROUPS BY THEIR FINITE QUOTIENTS

BY

M. R. BRIDSON*

*Mathematical Institute, University of Oxford
Andrew Wiles Building, ROQ, Woodstock Road, Oxford OX2 6GG, U.K
e-mail: bridson@maths.ox.ac.uk*

AND

M.D.E. CONDER**

*Department of Mathematics, University of Auckland
Private Bag 92019, Auckland 1142, New Zealand
e-mail: m.conder@auckland.ac.nz*

AND

A.W. REID†

*Department of Mathematics, University of Texas
Austin, TX 78712, USA
e-mail: areid@math.utexas.edu*

* Supported by a Senior Fellowship from the EPSRC of Great Britain and a Wolfson Research Merit Award from the Royal Society of London.

** Supported in part by the N.Z. Marsden Fund (grant UOA1015) and a James Cook Fellowship of the Royal Society of New Zealand.

† Supported in part by NSF grants.

Received October 25, 2013 and in revised form May 27, 2014

ABSTRACT

Let $\mathcal{C}(\Gamma)$ be the set of isomorphism classes of the finite groups that are quotients (homomorphic images) of Γ . We investigate the extent to which $\mathcal{C}(\Gamma)$ determines Γ when Γ is a group of geometric interest. If Γ_1 is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ and Γ_2 is a lattice in any connected Lie group, then $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ implies that $\Gamma_1 \cong \Gamma_2$. If F is a free group and Γ is a right-angled Artin group or a residually free group (with one extra condition), then $\mathcal{C}(F) = \mathcal{C}(\Gamma)$ implies that $F \cong \Gamma$. If $\Gamma_1 < \mathrm{PSL}(2, \mathbb{C})$ and $\Gamma_2 < G$ are non-uniform arithmetic lattices, where G is a semisimple Lie group with trivial centre and no compact factors, then $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ implies that $G \cong \mathrm{PSL}(2, \mathbb{C})$ and that Γ_2 belongs to one of finitely many commensurability classes. These results are proved using the theory of profinite groups; we do not exhibit explicit finite quotients that distinguish among the groups in question. But in the special case of two non-isomorphic triangle groups, we give an explicit description of finite quotients that distinguish between them.

1. Introduction

Let Γ be a finitely-generated group and let $\mathcal{C}(\Gamma)$ denote the set of isomorphism classes of finite groups that are quotients (homomorphic images) of Γ . If Γ is residually finite, then one can recover any finite portion of its Cayley graph or multiplication table by examining the finite quotients of the group. It is therefore natural to wonder whether, under reasonable hypotheses, the set $\mathcal{C}(\Gamma)$ might determine Γ up to isomorphism. (One certainly needs some hypotheses: for example, Remeslennikov [47] showed that a finitely-generated nilpotent group is not always uniquely determined by $\mathcal{C}(\Gamma)$.)

A celebrated instance of this question is the following (see Problem (F14) in [43]): If F_n is the free group of rank n , and Γ is a finitely-generated, residually finite group, then does $\mathcal{C}(\Gamma) = \mathcal{C}(F_n)$ imply that $\Gamma \cong F_n$? This question remains out of reach for the moment, as does the broader question of whether every Fuchsian group is distinguished from other finitely-generated, residually finite groups by its set of finite quotients. But in this paper we shall answer these questions in the affirmative for groups Γ that belong to various classes of groups that cluster naturally around Fuchsian groups. For example, we prove the following:

THEOREM 1.1: *Let Γ_1 be a finitely-generated Fuchsian group and let Γ_2 be a lattice in a connected Lie group. If $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$, then $\Gamma_1 \cong \Gamma_2$.*

We remind the reader that a Fuchsian group is, by definition, an infinite discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. To avoid trivial special cases, we shall assume that all of the groups considered are non-elementary (that is, are not virtually cyclic). Thus, for us, every Fuchsian group Γ has a subgroup of finite index that maps onto a non-abelian free group, and hence every finitely-generated group is a quotient of some finite-index subgroup of Γ . Deciding which groups arise as quotients of Γ itself is a more subtle matter, but much progress has been made on understanding the finite quotients, that is, deciphering the structure of $\mathcal{C}(\Gamma)$; see [17], [20], [31] and [37] and references therein.

The structure of the set $\mathcal{C}(\Gamma)$ is intimately connected with the subgroup growth of Γ , so it is interesting to contrast our results with what is known about the subgroup growth of Fuchsian groups (see [42] and [31]). Let $s_n(\Gamma)$ denote the number of subgroups in Γ that have index precisely n . In [42] an equivalence relation on the set of finitely-generated groups was introduced: Γ_1 and Γ_2 are declared to be equivalent if and only if $s_n(\Gamma_1) = (1 + o(1))s_n(\Gamma_2)$ asymptotically (that is, as $n \rightarrow \infty$). Also in [42], Müller and Schlage-Puchta exhibited an infinite sequence of pairwise non-isomorphic Fuchsian groups that are all equivalent in this sense. Our Theorem 1.1 shows that these equivalent groups are distinguished by their finite quotients.

Here, and throughout this paper, we use the term ‘surface group’ to mean a group that is isomorphic to the fundamental group of a closed surface of genus at least 1. A basic case in Theorem 1.1 is the situation where Γ_1 is a free group and Γ_2 is a surface group. This is not a difficult case to handle, but we give several proofs, each pointing to an argument that enables one to distinguish free groups by means of $\mathcal{C}(\Gamma)$ in broader classes; Theorem 1.2 records some of these classes and broader, more technically defined classes will be described in Section 4. One proof relies on the observation that in a surface group, every subgroup of finite index has even first betti number, whereas any finitely-generated free group has a subgroup of finite index with odd first betti number. A second proof relies on the observation that the number of elements needed to generate a surface group is equal to the rank of its abelianisation. A third proof relies on the fact that surface groups are good in the sense of Serre (see Section 2.4), LERF (locally extended residually finite), and of cohomological dimension 2 over a finite field.

THEOREM 1.2: *Let Γ_1 be a free group and let Γ_2 be a finitely-generated group. Then $\mathcal{C}(\Gamma_1) \neq \mathcal{C}(\Gamma_2)$ if Γ_2 satisfies one of the following conditions:*

- (a) Γ_2 is the fundamental group of a compact Kähler manifold;
- (b) Γ_2 is residually free and contains a surface subgroup;
- (c) Γ_2 is a non-free, right-angled Artin group.

In the preceding discussion, we have regarded finitely-generated free groups as examples of Fuchsian groups, residually free groups, right-angled Artin groups, and lattices in connected Lie groups. But except in the case of Fuchsian groups, we have not addressed the question of whether the operator \mathcal{C} distinguishes the isomorphism types within these classes. For general lattices, it certainly does not: it has long been known that there exist lattices in distinct nilpotent Lie groups that have the same collection of finite quotients [50] Chapter 11, and Aka [4] and [5] recently provided examples of such pairs of lattices in distinct semisimple Lie groups G_1, G_2 . Indeed there are examples where the real ranks of G_1 and G_2 are different (for example, rank 2 and rank 10) but nevertheless there exist lattices $\Gamma_j < G_j$ such that $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$. There also exist pairs of finitely-generated residually free groups $\Gamma_1 \not\cong \Gamma_2$ such that $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$; see Remark 4.19. In addition, in the context of 3-manifold groups, Funar [21] has shown the existence of non-homeomorphic torus bundles over the circle whose fundamental groups have the same finite quotients.

In contrast, there are reasons to suspect that for lattices in $\mathrm{PSL}(2, \mathbb{C})$, and in particular the fundamental groups of finite-volume orientable hyperbolic 3-manifolds, the situation may be more reminiscent of $\mathrm{PSL}(2, \mathbb{R})$, with lattices being uniquely determined by their finite quotients, both amongst themselves and amongst lattices in arbitrary Lie groups. In pursuit of this conviction, we prove the following theorem in Section 7.

THEOREM 1.3: *Let Γ_1 be a non-uniform lattice in $\mathrm{PSL}(2, \mathbb{C})$, and let Γ_2 be a non-uniform irreducible arithmetic lattice in a semisimple Lie group G that has trivial centre and no compact factors. If $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ then $G \cong \mathrm{PSL}(2, \mathbb{C})$. Moreover, if Γ_1 is arithmetic then the family of all Γ_2 with $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ divides into finitely many commensurability classes.*

It is interesting to compare the conclusion of Theorem 1.3 with Pickel's Theorem [45] and recent work by Aka [5]. In the latter it is shown that among higher rank lattices which have the Congruence Subgroup Property, there are only finitely many that can have the same set of finite quotients, whilst in the

former it is shown that only finitely many finitely-generated nilpotent groups can have the same set of finite quotients.

We have deliberately stated our results in terms of the naively defined set $\mathcal{C}(\Gamma)$, but it is both natural and useful to regard them as statements about the profinite completions of the groups in question. We remind the reader that the **profinite completion** $\widehat{\Gamma}$ of a group Γ is the inverse limit of the finite quotients Γ/N of Γ . (The maps in the inverse system are the obvious ones: if $N_1 < N_2$ then $\Gamma/N_1 \rightarrow \Gamma/N_2$.) If Γ is finitely-generated, then $\mathcal{C}(\Gamma) = \mathcal{C}(\widehat{\Gamma})$; thus $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ implies $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$. Less obviously, $\mathcal{C}(\Gamma)$ uniquely determines $\widehat{\Gamma}$ (see [19], [48] p. 89 and Section 3 below). Thus we may rephrase all of the preceding results in terms of profinite completions. For example:

THEOREM 1.4: *Let Γ_1 and Γ_2 be as in Theorem 1.1 or 1.2. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ then $\Gamma_1 \cong \Gamma_2$.*

Our invocation of profinite groups is more than a matter of terminology: in particular, our proofs of Theorems 1.1 and 1.3 rely on various aspects of the theory of profinite groups, and the interplay between the abstract group and its profinite completion. Certain of the arguments also rely on deep properties of the lattices concerned.

Our detours through the theory of profinite groups are such that, for the most part, we prove that groups have distinct profinite completions *without exhibiting specific finite quotients that distinguish them*. Indeed, even in relatively small examples, identifying such finite quotients appears to be rather delicate. In the final section of this paper we tackle this problem head-on in the case of triangle groups, where it succumbs to a direct analysis.

We close this introduction with some comments concerning logic and decidability. First, note that whenever one has a class of finitely-presented residually finite groups where $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ implies $\Gamma_1 \cong \Gamma_2$, one obtains a solution to the isomorphism problem in that class of groups. Indeed, since each group is finitely-presented, one can effectively enumerate its finite quotients, and in this way one will prove in a finite number of steps that $\mathcal{C}(\Gamma_1) \neq \mathcal{C}(\Gamma_2)$ if this is the case; and running this partial algorithm in parallel with a naive search for mutually-inverse homomorphisms $\Gamma_1 \leftrightarrow \Gamma_2$ will (in finite time) determine whether or not Γ_1 is isomorphic to Γ_2 . Similarly, Theorem 1.1 implies that there exists an algorithm that, given a finite presentation of a lattice in a connected Lie group, will determine whether or not the group presented is Fuchsian. And

Theorem 1.3 provides an algorithm for determining if a finite presentation of a non-uniform arithmetic lattice is Kleinian.

A recent result of Jarden and Lubotzky [26], which relies on deep work of Nikolov and Segal [44], shows that there exists an isomorphism $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ if and only if there is an equivalence of the elementary theories $\text{Th}(\widehat{\Gamma}_1) \equiv \text{Th}(\widehat{\Gamma}_2)$ (in the sense of first order logic). So our results can also be interpreted as criteria for determining if such elementary theories are distinct. This is particularly striking in the light of the solution to Tarski's problem by Sela [51] and Kharlampovich–Myasnikov [28], which implies that the fundamental group of any closed surface of Euler characteristic at most -2 has the same elementary theory as a free group of any finite rank $r \geq 2$. For clarity, we highlight a special case of this discussion, but stress that we have contributed only a small part of the proof.

THEOREM 1.5: *If Γ_1 and Γ_2 are any two non-isomorphic, non-elementary, torsion-free Fuchsian groups, then $\text{Th}(\Gamma_1) \equiv \text{Th}(\Gamma_2)$ but $\text{Th}(\widehat{\Gamma}_1) \not\equiv \text{Th}(\widehat{\Gamma}_2)$.*

We conclude this introduction with a brief outline of the paper. In Section 2 we recall some basic properties of the profinite completion of a (discrete) residually finite group Γ , the correspondence between subgroups of Γ and that of its profinite completion, as well as the relationship between the profinite completion of subgroups and their closures in the profinite completion of Γ . In addition, we recall some of the theory of the cohomology of profinite groups, including Serre's notion of goodness and cohomological dimension, and we discuss actions of profinite groups on profinite trees. In Section 3 we describe a connection (via Lück's Approximation Theorem) between groups with isomorphic profinite completions and the first L^2 -Betti number. In Section 4 we prove a variety of results that restrict the class of groups that can have the same profinite completion as a free group, and in Section 5 we prove a result controlling torsion in the profinite completion of a finitely-generated Fuchsian group. The proof of Theorem 1.1 is completed in Section 6, by putting together results proved in Sections 2, 3 and 5. In Section 7 we prove Theorem 1.3, relying heavily on Section 2. In Section 8 we explain how to distinguish triangle groups more directly, by describing exhibit explicit finite quotients that do so.

ACKNOWLEDGEMENTS. We thank Gopal Prasad and Alex Lubotzky for helpful comments concerning Theorem 3.6. The first and third authors thank the University of Auckland, Massey University and The New Zealand Institute for

Advanced Study for supporting the visit during which this collaboration began. They particularly thank Gaven Martin for his hospitality during that visit. They also thank the Mittag-Leffler Institute for its hospitality during the final drafting of this paper.

2. Profinite completions

In this section we recall some background on profinite completions and the theory of profinite groups; see [48], [50] and [52] for more details.

2.1. RESIDUALLY FINITE GROUPS AND THEIR COMPLETIONS. By definition, the **profinite completion** $\widehat{\Gamma}$ of a finitely-generated group Γ is the inverse limit of the system of finite quotients of Γ . The natural map $\Gamma \rightarrow \widehat{\Gamma}$ is injective if and only if Γ is residually finite. By definition $\widehat{\Gamma}$ maps onto every finite quotient of Γ . On the other hand, the image of Γ is dense regardless of whether Γ is residually finite, so the restriction to Γ of any continuous homomorphism from $\widehat{\Gamma}$ to a discrete group (such as a finite group) is surjective. A deep theorem of Nikolov and Segal [44] states that every homomorphism from $\widehat{\Gamma}$ to a finite group is continuous, so we have the following basic result (in which $\text{Hom}(G, Q)$ denotes the set of homomorphisms from the group G to the group Q , and $\text{Epi}(G, Q)$ denotes the set of epimorphisms).

LEMMA 2.1: *Let Γ be a finitely-generated group and let $\iota : \Gamma \rightarrow \widehat{\Gamma}$ be the natural map to its profinite completion. Then, for every finite group Q , the map $\text{Hom}(\widehat{\Gamma}, Q) \rightarrow \text{Hom}(\Gamma, Q)$ defined by $g \mapsto g \circ \iota$ is a bijection, and this restricts to a bijection $\text{Epi}(\widehat{\Gamma}, Q) \rightarrow \text{Epi}(\Gamma, Q)$.*

COROLLARY 2.2: *If Γ_1 is finitely-generated and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $|\text{Hom}(\Gamma_1, Q)| = |\text{Hom}(\Gamma_2, Q)|$ for every finite group Q .*

Closely related to this we have the following proposition (which is taken from [48, Chapter 3.2], using the Nikolov–Segal theorem to replace ‘open’ by ‘finite-index’).

Notation: Given a subset X of a profinite group G , we write \overline{X} to denote the closure of X in G .

PROPOSITION 2.3: *With Γ as above, there is a one-to-one correspondence between the set \mathcal{X} of finite-index subgroups of Γ and the set \mathcal{Y} of all finite-index*

subgroups of $\widehat{\Gamma}$. If Γ is identified with its image in $\widehat{\Gamma}$, then this bijection takes $H \in \mathcal{X}$ to \overline{H} , and conversely, takes $Y \in \mathcal{Y}$ to $Y \cap \Gamma$. Also $|\Gamma : H| = |\widehat{\Gamma} : \overline{H}|$. Moreover, \overline{H} is normal in $\widehat{\Gamma}$ if and only if H is normal in Γ , in which case $\Gamma/H \cong \widehat{\Gamma}/\overline{H}$.

COROLLARY 2.4: *Let Γ be a finitely-generated group, and for each $d \in \mathbb{N}$, let M_d denote the intersection of all normal subgroups of index at most d in Γ . Then the closure $\overline{M_d}$ of M_d in $\widehat{\Gamma}$ is the intersection of all normal subgroups of index at most d in $\widehat{\Gamma}$, and hence $\bigcap_{d \in \mathbb{N}} \overline{M_d} = 1$.*

Proof. If N_1 and N_2 are the kernels of epimorphisms from Γ to finite groups Q_1 and Q_2 , then $\overline{N_1 \cap N_2}$ is the kernel of the extension of $\Gamma \rightarrow Q_1 \times Q_2$ to $\widehat{\Gamma}$, while $\overline{N_1} \times \overline{N_2}$ is the kernel of the map $\widehat{\Gamma} \rightarrow Q_1 \times Q_2$ that one gets by extending each of $\Gamma \rightarrow Q_i$ and then taking the direct product. The uniqueness of extensions tells us that these maps coincide, and hence $\overline{N_1 \cap N_2} = \overline{N_1} \cap \overline{N_2}$. The claims follow from repeated application of this observation. \blacksquare

We finish this subsection with a discussion of the relationship between the statement $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ for finitely-generated residually finite groups Γ_1 and Γ_2 , and the statement that the profinite completions $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic.

Suppose that $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic. Following the aforementioned work of Nikolov and Segal [44], by ‘isomorphic’ we simply mean ‘isomorphic as groups’, for by [44], each group-theoretic isomorphism is continuous, and so any such isomorphism will be an isomorphism of topological groups. Given an isomorphism between $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, it is easy to deduce from Proposition 2.3 that this implies that Γ_1 and Γ_2 have the same collection of finite quotient groups. The converse is also true; see [19] and [48, pp. 88–89].

THEOREM 2.5: *Suppose that Γ_1 and Γ_2 are finitely-generated abstract groups. Then $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic if and only if $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

We close the subsection with a comment on terminology. In [24] (see also [5]) the notion of genus of a group Γ from within a class of groups \mathcal{G} is discussed: it is defined to be the set of isomorphism classes of groups in \mathcal{G} with the same profinite completion as Γ , and is denoted by $\mathbf{g}(\mathcal{G}, \Gamma)$. Hence if we let \mathcal{L} denote the set of all lattices in connected Lie groups, our main result can be rephrased as:

THEOREM 2.6: *Let Γ be a finitely-generated Fuchsian group, then $\mathbf{g}(\mathcal{L}, \Gamma) = \{\Gamma\}$.*

We have chosen not to use this notation elsewhere in this article in order to avoid confusion with the classical use of the term genus in the sense of surfaces (and their fundamental groups).

2.2. COMPLETION AND CLOSURE. Let Γ be a residually finite (abstract) group, and $u : H \hookrightarrow \Gamma$ the inclusion mapping of a proper subgroup H of Γ . The canonical homomorphism $\hat{u} : \hat{H} \rightarrow \hat{\Gamma}$ is injective if and only if *the profinite topology on Γ induces the full profinite topology on H* ; in more elementary terms, for every normal subgroup $H_1 < H$ of finite index, there is a normal subgroup $\Gamma_1 < \Gamma$ of finite index with $\Gamma_1 \cap H < H_1$.

Recall that if Γ is a group and H a subgroup of Γ , then Γ is called *H -separable* if for every $g \in G \setminus H$, there is a subgroup K of finite index in Γ such that $H \subset K$ but $g \notin K$; equivalently, the intersection of all finite index subgroups in Γ containing H is precisely H . The group Γ is called **LERF** (or **subgroup separable**) if it is H -separable for every finitely-generated subgroup H , or equivalently, if every finitely-generated subgroup is a closed subset in the profinite topology.

In the context of this paper, it is important to note that even if the subgroup H of Γ is separable, it need not be the case that the profinite topology on Γ induces the full profinite topology on H . Stronger separability properties do suffice, however, as we now indicate. The following lemma is well-known, but we include a proof for the convenience of the reader.

LEMMA 2.7: *Let Γ be a finitely-generated group, and H a finitely-generated subgroup of Γ . Suppose that Γ is H_1 -separable for every finite index subgroup H_1 in H . Then the profinite topology on Γ induces the full profinite topology on H ; that is, the natural map $\hat{H} \rightarrow \overline{H}$ is an isomorphism.*

Proof. Since Γ is H_1 separable, the intersection of all subgroups of finite index in Γ containing H_1 is H_1 itself. Thus we can find a finite-index subgroup K_1 of Γ such that $K_1 \cap H = H_1$. Replacing K_1 by a normal subgroup Γ_1 of finite index provides the required subgroup $\Gamma_1 \cap H$ of H_1 . ■

Subgroups of finite index are obviously separable.

COROLLARY 2.8: *If Γ is residually finite and H is a finite-index subgroup of Γ , then the natural map from \widehat{H} to \overline{H} (the closure of H in $\widehat{\Gamma}$) is an isomorphism.*

COROLLARY 2.9: *If Γ_1 and Γ_2 are finitely-generated and residually finite and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then there is a bijection I from the set of subgroups of finite index in Γ_1 to the set of subgroups of finite-index in Γ_2 , such that $\widehat{I(H)} \cong \widehat{H}$ and $|\Gamma_1 : H| = |\Gamma_2 : I(H)|$ for every subgroup H of finite index in Γ_1 . Moreover, H is normal in Γ_1 if and only if $I(H)$ is normal in Γ_2 .*

Proof. If $\phi : \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$ is an isomorphism, define $I(H) = \phi(\overline{H}) \cap \Gamma_2$. Proposition 2.3 tells us that I is a bijection and that $|\Gamma_1 : H| = |\Gamma_2 : I(H)|$. Corollary 2.8 tells us that $\widehat{H} \cong \overline{H} \cong \widehat{I(H)} \cong \widehat{I(H)}$. ■

2.3. BETTI NUMBERS. The first betti number of a finitely-generated group is

$$b_1(\Gamma) = \text{rank}((\Gamma/[\Gamma, \Gamma]) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

This invariant can be detected in the finite quotients of Γ since it is the *greatest integer b such that Γ has the elementary p -group of rank b among its quotients, for every prime p* . We exploit this observation as follows:

LEMMA 2.10: *Let Λ and Γ be finitely-generated groups. If Λ is isomorphic to a dense subgroup of $\widehat{\Gamma}$, then $b_1(\Lambda) \geq b_1(\Gamma)$.*

Proof. For every finite group A , each epimorphism $\widehat{\Gamma} \rightarrow A$ will restrict to an epimorphism from both Γ and Λ , and every epimorphism $\Gamma \rightarrow A$ extends to an epimorphism $\widehat{\Gamma} \rightarrow A$. Therefore, if Γ maps onto an elementary p -group of rank b , then so does Λ (but perhaps not vice versa). ■

COROLLARY 2.11: *If Γ_1 and Γ_2 are finitely-generated and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $b_1(\Gamma_1) = b_1(\Gamma_2)$.*

2.4. GOODNESS AND COHOMOLOGICAL DIMENSION. Following Serre [52], we say that a group Γ is **good** if for every finite Γ -module M , the homomorphism of cohomology groups $H^n(\widehat{\Gamma}; M) \rightarrow H^n(\Gamma; M)$ induced by the natural map $\Gamma \rightarrow \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$. (See [52] and [48, Chapter 6] for details about the cohomology of profinite groups.)

It is easy to see that free groups are good, but it seems that goodness is hard to establish in general. One can, however, establish goodness for a group Γ that

is LERF if one has a well-controlled splitting of the group as a graph of groups [23]. Using the fact that Fuchsian groups are LERF [49], it is proved in [23] that Fuchsian groups are good.

Now a useful criterion for goodness is provided by the next lemma due to Serre (see [52, Chapter 1, Section 2.6])

LEMMA 2.12: *The group Γ is good if there is a short exact sequence*

$$1 \rightarrow N \rightarrow \Gamma \rightarrow H \rightarrow 1,$$

such that H and N are good, N is finitely-generated, and the cohomology group $H^q(N, M)$ is finite for every q and every finite Γ -module M .

Recent work of Agol [1] and Wise [57] establishes that finite-volume hyperbolic 3-manifolds are virtually fibered, and so Lemma 2.12 can be applied to establish goodness for the fundamental groups of such manifolds. In addition, since goodness is preserved by commensurability (see [23] Lemma 3.2), it follows that the fundamental groups of all finite-volume hyperbolic 3-manifolds are good. We summarize this discussion, and that for Fuchsian groups, in the following result:

THEOREM 2.13: *Lattices in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ are good.*

Let G be a profinite group. Then the **p -cohomological dimension** of G is the least integer n such that for every discrete torsion G -module A and for every $q > n$, the p -primary component of $H^q(G; A)$ is zero, and this is denoted by $\mathrm{cd}_p(G)$. The cohomological dimension of G is defined as the supremum of $\mathrm{cd}_p(G)$ over all primes p , and this is denoted by $\mathrm{cd}(G)$.

We also retain the standard notation $\mathrm{cd}(\Gamma)$ for the cohomological dimension (over \mathbb{Z}) of a *discrete* group Γ .

LEMMA 2.14: *Let Γ be a discrete group that is good. If $\mathrm{cd}(\Gamma) \leq n$, then $\mathrm{cd}(\widehat{\Gamma}) \leq n$.*

Discrete groups of finite cohomological dimension (over \mathbb{Z}) are torsion-free. We are interested in conditions that allow one to deduce that $\widehat{\Gamma}$ is also torsion-free. For this we need the following result that mirrors the behaviour of cohomological dimension for discrete abstract groups (see [52, Chapter 1 §3.3]).

PROPOSITION 2.15: *Let p be a prime, let G be a profinite group, and H a closed subgroup of G . Then $\text{cd}_p(H) \leq \text{cd}_p(G)$.*

COROLLARY 2.16: *Suppose that Γ is a residually finite, good group of finite cohomological dimension over \mathbb{Z} . Then $\widehat{\Gamma}$ is torsion-free.*

Proof. If $\widehat{\Gamma}$ were not torsion-free, then it would have an element x of prime order, say q . Since $\langle x \rangle$ is a closed subgroup of $\widehat{\Gamma}$, Proposition 2.15 tells us that $\text{cd}_p(\langle x \rangle) \leq \text{cd}_p(\widehat{\Gamma})$ for all primes p . But $H^{2k}(\langle x \rangle; \mathbf{F}_q) \neq 0$ for all $k > 0$, so $\text{cd}_q(\langle x \rangle)$ and $\text{cd}_q(\widehat{\Gamma})$ are infinite. Since Γ is good and has finite cohomological dimension over \mathbb{Z} , we obtain a contradiction from Lemma 2.14. ■

The following simple observation will prove very useful in the sequel.

COROLLARY 2.17: *Let Γ_1 and Γ_2 be finitely-generated (abstract) residually finite groups with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Assume that Γ_1 is good and $\text{cd}(\Gamma_1) = n < \infty$. Furthermore, assume that H is a good subgroup of Γ_2 for which the natural mapping $\widehat{H} \rightarrow \widehat{\Gamma}_2$ is injective. Then $H^q(H; \mathbf{F}_p) = 0$ for all $q > n$ and all primes p .*

Proof. If $H^q(H; \mathbf{F}_p)$ were non-zero for some $q > n$, then by goodness we would have $H^q(\widehat{H}; \mathbf{F}_p) \neq 0$, so $\text{cd}_p(\widehat{H}) \geq q > n$. Now $\widehat{H} \rightarrow \widehat{\Gamma}_2$ is injective and so $\widehat{H} \cong \overline{H}$. Hence $\widehat{\Gamma}_1$ contains a closed subgroup of p -cohomological dimension greater than n , a contradiction. ■

2.5. ACTIONS ON PROFINITE TREES. In this subsection we recall some basics of Bass-Serre theory for profinite groups that will be required later. (See [41] and [48] for more detailed accounts.)

If G_1 and G_2 are profinite groups with a common profinite subgroup H , then we denote by $G_1 \amalg_H G_2$ the profinite amalgamated free product of G_1 and G_2 , with H amalgamated.

As in the case of abstract groups, if a profinite group G splits as a profinite amalgamated free product, then there is control on where any given finite subgroup lies. The following result was proved for free profinite products in [25] and for more general profinite amalgamated free products in [41, Theorem 3.10].

THEOREM 2.18: *Let $G = G_1 \amalg_H G_2$ be any profinite amalgamated free product. If K is a finite subgroup of G , then K is conjugate to a subgroup of G_1 or G_2 .*

Now if A and B are abstract groups with a common subgroup C , and Γ denotes the abstract amalgamated free product $A *_C B$, then it is not generally true that $\widehat{\Gamma}$ is isomorphic to $\widehat{A} \amalg_{\widehat{C}} \widehat{B}$. But this does hold when Γ is residually finite and the profinite topology on Γ induces the full profinite topologies on A , B and C ; see [48, pp. 379–380]. In particular, we record the following consequence that will be useful for us.

PROPOSITION 2.19: *If $\Gamma = \Gamma_1 *_C \Gamma_2$ is LERF, then $\widehat{\Gamma} \cong \widehat{\Gamma}_1 \amalg_{\widehat{C}} \widehat{\Gamma}_2$.*

A case of particular interest to us occurs when Γ is a finitely-generated Fuchsian group; such groups are known to be LERF (see [49]).

COROLLARY 2.20: *If $\Gamma = A *_C B$ is a finitely-generated Fuchsian group, then $\widehat{\Gamma} \cong \widehat{A} \amalg_{\widehat{C}} \widehat{B}$.*

Note that it follows from [32] that if Γ is any finitely-generated Fuchsian group that is not a (p, q, r) -triangle group, then Γ can be decomposed as non-trivial free products with amalgamation $A *_C B$. (For most Fuchsian groups, this follows easily using a geometric decomposition of the hyperbolic 2-orbifold.)

3. L^2 -Betti numbers and profinite completions

L^2 -Betti numbers provide powerful invariants for distinguishing the profinite completions of various classes of groups that cluster around free groups. We refer the reader to Lück’s treatise [36] for a comprehensive introduction to L^2 -Betti numbers. For our purposes, it is best to view these invariants not in terms of their original (more analytic) definition, but instead as asymptotic invariants of towers of finite-index subgroups. This is made possible by the Lück’s Approximation Theorem [35]:

THEOREM 3.1: *Let G be a finitely-presented group, and let $G = G_1 > G_2 > \dots > G_m > \dots$ be a sequence of finite-index subgroups that are normal in G and intersect in the identity. The first L^2 -Betti number of G is given by the formula*

$$b_1^{(2)}(G) = \lim_{m \rightarrow \infty} \frac{b_1(G_m)}{|G : G_m|}.$$

An important point to note is that this limit does not depend on the tower, and hence is an invariant of G .

PROPOSITION 3.2: *Let Λ and Γ be finitely-presented residually finite groups and suppose that Λ is a dense subgroup of $\widehat{\Gamma}$. Then $b_1^{(2)}(\Gamma) \leq b_1^{(2)}(\Lambda)$.*

Proof. For each positive integer d let M_d be the intersection of all normal subgroups of index at most d in Γ , and let $L_d = \Lambda \cap \overline{M}_d$ in $\widehat{\Gamma}$. We saw in Corollary 2.4 that $\bigcap_d \overline{M}_d = 1$, and so $\bigcap_d L_d = 1$. Since Λ and Γ are both dense in $\widehat{\Gamma}$, the restriction of $\widehat{\Gamma} \rightarrow \widehat{\Gamma}/\overline{M}_d$ to each of these subgroups is surjective, and hence

$$|\Lambda : L_d| = |\widehat{\Gamma} : \overline{M}_d| = |\Gamma : M_d|.$$

Now L_d is dense in \overline{M}_d , while $\widehat{M}_d = \overline{M}_d$, so Lemma 2.10 implies that $b_1(L_d) \geq b_1(M_d)$, and then we can use the towers (L_d) in Λ and (M_d) in Γ to compare L^2 -betti numbers and find

$$b_1^{(2)}(\Gamma) = \lim_{d \rightarrow \infty} \frac{b_1(M_d)}{|\Gamma : M_d|} \leq \lim_{d \rightarrow \infty} \frac{b_1(L_d)}{|\Lambda : L_d|} = b_1^{(2)}(\Lambda),$$

by Lück’s approximation theorem. ■

This has the following important consequence:

COROLLARY 3.3: *Let Γ_1 and Γ_2 be finitely-presented residually finite groups. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $b_1^{(2)}(\Gamma_1) = b_1^{(2)}(\Gamma_2)$.*

Remark 3.4: In [12] the first and third authors prove versions of Proposition 3.2 and Corollary 3.3 for finitely-presented groups which are residually- p for some prime p . In this setting the profinite completion is replaced by the pro- p completion.

When we make further use of L^2 -betti numbers, we will exploit the following additional elementary observation:

PROPOSITION 3.5: *If Γ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ with rational Euler characteristic $\chi(\Gamma)$, then $b_1^{(2)}(\Gamma) = -\chi(\Gamma)$.*

Proof. It follows from Lück’s approximation theorem that if H is a subgroup of index d in Γ (which is finitely-presented) then $b_1^{(2)}(H) = d b_1^{(2)}(\Gamma)$. Rational Euler characteristic is multiplicative in the same sense. Thus we may pass to a torsion-free subgroup of finite index in Γ , and assume that it is either a free group F_r of rank r , or the fundamental group Σ_g of a closed orientable surface of genus g . In both cases, a subgroup Γ_d of index d in Γ will be a group of the same form, with the rank and genus given by the expressions $d(r - 1) + 1$

and $d(g - 1) + 1$. In the free case, the (ordinary) first betti number is $d(r - 1) + 1$ and so $b_1(\Gamma_d) = 1 - d \chi(\Gamma)$, while in the second case the first betti number is $2d(g - 1) + 2$ and so $b_1(\Gamma_d) = 2 - d \chi(\Gamma)$. In both cases, dividing by $d = |\Gamma : \Gamma_d|$ and taking the limit, we find $b_1^{(2)}(\Gamma) = -\chi(\Gamma)$. ■

We shall now present a reduction of Theorem 1.1 using Corollary 3.3 and Proposition 3.5. The proof of Theorem 1.1 will be completed in Sections 5 and 6.

We draw the reader’s attention to the fact that in Corollary 3.3 both groups are assumed to be residually finite, whereas in the following theorem Λ need not be residually finite since we have not assumed that the Lie group G is linear. The proof shows how to by-pass this problem.

THEOREM 3.6: *Let Λ be a lattice in a connected Lie group G and suppose that $\widehat{\Lambda} \cong \widehat{\Gamma}$ for some Fuchsian group Γ . Then, either Λ is isomorphic to a Fuchsian group, or else $\widehat{\Lambda}$ has a non-trivial finite normal subgroup.*

Proof. Let $Z(G)$ denote the centre of G and let $Z = \Lambda \cap Z(G)$. As $Z(G)$ is the kernel of the adjoint representation, $G/Z(G)$ is linear. In addition, since Λ is a lattice in a connected Lie group it is finitely generated (see [46] §6.18), and it follows that Λ/Z is a finitely generated residually finite group. In particular, Z contains the kernel of the natural map $\Lambda \rightarrow \widehat{\Lambda}$, which we call I . We first claim that either $Z = I$ or else $\Delta := \Lambda/I$ has a non-trivial finite normal subgroup; at the end of this proof we shall show that such a subgroup remains normal in $\widehat{\Delta}$, which is isomorphic to $\widehat{\Lambda}$ because Δ is the image of Λ in $\widehat{\Lambda}$. The only other possibility is that Z/I is an infinite central subgroup of the residually finite group Δ . But Δ cannot have an infinite normal amenable subgroup, because Cheeger–Gromov [15] prove that this would force the first L^2 -betti number of Δ to be zero, contradicting Corollary 3.3 and Proposition 3.5, since $\widehat{\Delta} \cong \widehat{\Gamma}$.

Now we deal with the case $I = Z$, adapting an argument of Lott [34]. Let Rad be the radical of G , let K be the maximal compact connected normal subgroup of a Levi complement, let $G_1 = \text{Rad} \cdot K$ and recall that $G_2 = G/G_1$ is a connected semisimple Lie group with no compact connected normal subgroup. The short exact sequence

$$1 \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow 1$$

restricts to

$$1 \rightarrow \Lambda_1 \rightarrow \Lambda \rightarrow \Lambda_2 \rightarrow 1,$$

where $\Lambda_1 = \Lambda \cap \widehat{G}_1$, and factoring out Z (and its image Z' in Λ_2) we get

$$1 \rightarrow \Lambda_1 Z/Z \rightarrow \Lambda/Z \rightarrow \Lambda_2/Z' \rightarrow 1.$$

By repeating the argument with L^2 -betti numbers, we see that $\Lambda_1 Z/Z$ must be finite, as it is amenable (since Λ_1 is a closed subgroup of the amenable G_1) and $\Delta = \Lambda/Z$ has the same profinite completion as the Fuchsian group Γ . So either we again obtain a finite non-trivial normal subgroup in Δ , or else $\Lambda_1 Z/Z$ is trivial, in which case $\Delta \cong \Lambda_2/Z'$. Now we appeal to the fact that since Λ_2 is a lattice in a *semisimple* Lie group with no compact factors, Λ_2 modulo its centre $Z(\Lambda_2)$ is itself a lattice in a semisimple Lie group (see [46] Corollary 8.27). If $Z' = Z(\Lambda_2)$, then $\Delta = \Lambda_2/Z(\Lambda_2)$ and we are in the setting considered by Lott ([34], Lemma 1), who proves that either Δ has a non-trivial finite normal subgroup or else Δ is isomorphic to a Fuchsian group. If Z' is a proper subgroup of $Z(\Lambda_2)$ then Δ has a non-trivial centre and we conclude as before.

All that remains is to prove that if $M < \Delta$ is a non-trivial normal subgroup, then M remains normal in $\widehat{\Delta}$. Indeed for any group H , if N is a finite normal subgroup of H , then the image of N in \widehat{H} , which we denote \overline{N} , is also normal. For if it were not normal then there would be $n \in \overline{N}$ and $x \in \widehat{H}$ such that the set $S = \{xnx^{-1}n' \mid n' \in \overline{N}\}$ did not contain the identity. By residual finiteness, the finite set $S \cup \{1\}$ would inject into some finite quotient of \widehat{H} . But H maps onto this finite quotient, and N is normal in H . This contradiction shows that \overline{N} is normal in \widehat{H} , as claimed. ■

Remark 3.7: In Section 5 we shall conclude that the second possibility in Theorem 3.6 does not occur, thereby reducing the proof of Theorem 1.1 to Fuchsian groups.

4. Three obstructions to profinite freeness

In this section we present three different proofs of the fact that the fundamental group of a closed surface cannot have the same set of finite quotients as a free group. Each proof highlights a different obstruction to having the same profinite completion as a free group, and thereby sheds light on the fundamental question

of whether a finitely-generated residually finite group that has the same profinite completion as a free group must itself be free.

4.1. THE HOPF PROPERTY AND RANK. It is well-known and easy to prove that if a finitely-generated group Γ is residually finite, then it has the Hopf property — that is, every epimorphism $\Gamma \rightarrow \Gamma$ is an isomorphism. Certain proofs of this fact extend in a straightforward manner to finitely-generated profinite groups. The following lemma captures this idea in more pedestrian language.

LEMMA 4.1: *Let $\phi : \Gamma_1 \rightarrow \Gamma_2$ be an epimorphism of finitely-generated groups. If Γ_1 is residually finite and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then ϕ is an isomorphism.*

Proof. Let $k \in \ker \phi$. If k were non-trivial, then since Γ_1 is residually finite, there would be a finite group Q and an epimorphism $f : \Gamma_1 \rightarrow Q$ such that $f(k) \neq 1$. This map f does not lie in the image of the injection $\text{Hom}(\Gamma_2, Q) \hookrightarrow \text{Hom}(\Gamma_1, Q)$ defined by $g \mapsto g \circ \phi$. Thus $|\text{Hom}(\Gamma_1, Q)| > |\text{Hom}(\Gamma_2, Q)|$, contradicting Corollary 2.2. ■

Definition 4.2: The **rank** $d(\Gamma)$ of a finitely-generated group Γ is the least integer k such that Γ has a generating set of cardinality k . The **rank** $\widehat{d}(G)$ of a profinite group G is the least integer k for which there is a subset $S \subset G$ with $k = |S|$ and $\langle S \rangle$ is dense in G .

In the following proposition, we do not assume that Γ is residually finite.

PROPOSITION 4.3: *Let Γ be a finitely-generated group and let F_n be a free group. If Γ has a finite quotient Q such that $d(\Gamma) = d(Q)$, and $\widehat{\Gamma} \cong \widehat{F}_n$, then $\Gamma \cong F_n$.*

Proof. First $\widehat{\Gamma} \cong \widehat{F}_n$, so Q is a quotient of F_n . Hence $n \geq d(Q)$. But $d(Q) = d(\Gamma)$ and for every integer $s \geq d(\Gamma)$ there exists an epimorphism $F_s \rightarrow \Gamma$. Thus we obtain an epimorphism $F_n \rightarrow \Gamma$, and application of the preceding lemma completes the proof. ■

COROLLARY 4.4: *Let Γ be a finitely-generated group. If Γ and its abelianisation have the same rank, then $\widehat{\Gamma} \cong \widehat{F}_n$ if and only if $\Gamma \cong F_n$.*

Proof. Every finitely-generated abelian group A has a finite quotient of rank $d(A)$. ■

Let K be a finite simplicial graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E \subset V \times V$. Then the **right angled Artin group** (or **RAAG**) associated with K is the group $A(K)$ given by the following presentation:

$$A(K) = \langle v_1, \dots, v_n \mid [v_i, v_j] = 1 \text{ for all } i, j \text{ such that } \{v_i, v_j\} \in E \rangle.$$

For example, if K is a graph with n vertices and no edges, then $A(K)$ is the free group of rank n , while if K is the complete graph on n vertices, then $A(K)$ is the free abelian group \mathbb{Z}^n of rank n .

If the group Γ has a presentation of the form $\langle A \mid R \rangle$ where A is finite and all of the relators $r \in R$ lie in the commutator subgroup of the free group $F(A)$, then both Γ and its abelianisation (which is free abelian) have rank $|A|$. The standard presentations of RAAGs and closed surface groups are of this form. Also the fundamental group of a closed non-orientable surface has a presentation of the form $\Gamma = \langle a_1, b_1, \dots, a_g, b_g, c \mid c^2 = \prod_i [a_i, b_i] \rangle$, so again Γ has the same rank as its abelianisation.

PROPOSITION 4.5: *If Γ is a right-angled Artin group that is not free, or the fundamental group of a closed surface, then there exists no free group F such that $\widehat{F} \cong \widehat{\Gamma}$.*

Remark 4.6: Proposition 4.3 shows that if Γ is a finitely-generated, residually finite group that is not free, but $\widehat{\Gamma}$ is a free profinite group, then $d(\Gamma) > \widehat{d}(\widehat{\Gamma})$. In this context, it is worth noting that for an arbitrarily large integer d , there exist residually finite hyperbolic groups Γ such that $d(\Gamma) = d$ but $\widehat{d}(\widehat{\Gamma}) = 2$. To construct such groups, one can follow the first steps of the main construction in [11]. For each integer $d \geq 3$ there exists a finitely-presented group Q with $d(Q) = d$ such that $\widehat{Q} = \{1\}$ and $H_2(Q, \mathbb{Z}) = 0$. The Rips construction provides a short exact sequence $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$ such that Γ satisfies a strict small-cancellation condition and N is a 2-generator group. Wise [56] proves that the hyperbolic group Γ is the fundamental group of a compact non-positively curved cube complex, and Agol [1] proves that such groups are linear, hence residually-finite. It is shown in [11] that $\widehat{N} \cong \widehat{\Gamma}$, from which it follows that $\widehat{d}(\widehat{\Gamma}) = 2$. But Γ maps onto Q , and therefore $d(\Gamma) \geq d$.

4.2. THE PARITY OF VIRTUAL BETTI NUMBERS.

LEMMA 4.7: *Let F be a free group of rank $r \geq 1$. If $\widehat{\Gamma} \cong \widehat{F}$, then for every positive integer d there exists a subgroup Λ of index d in Γ such that $b_1(\Lambda) =$*

$d(r - 1) + 1$. In particular, Γ has subgroups of finite index with odd first betti number.

Proof. First, F has subgroups of every possible index, and every subgroup of index d is free of rank $d(r - 1) + 1$. Next, by Corollary 2.9, for every subgroup $H < F$ of index d there is a subgroup $\Lambda < \Gamma$ of index d such that $\widehat{H} \cong \widehat{\Lambda}$, and then by Lemma 2.11 we have $b_1(\Lambda) = b_1(H) = d(r - 1) + 1$. ■

The first betti number of the fundamental group of a closed orientable surface is even, and any subgroup of finite index in such a group is again the fundamental group of a closed orientable surface. Thus Lemma 4.7 provides us with a second proof that surface groups do not have the same profinite completions as free groups. This argument can be extended to the fundamental groups of compact Kähler manifolds. Recall that a smooth manifold X with a Riemannian metric g , symplectic form ω , and complex structure J is said to be **Kähler** if $g(J(u), J(v)) = g(u, v)$ and $g(J_p(u), v) = \omega(u, v)$ for all $p \in X$ and all tangent vectors $u, v \in T_p(X)$. Although every finitely-presented group occurs as the fundamental group of a closed symplectic manifold and a closed complex manifold, the Kähler condition imposes significant constraints (see [6]). For example:

PROPOSITION 4.8: *If Γ is the fundamental group of a compact Kähler manifold, then there exists no non-trivial free group F such that $\widehat{F} \cong \widehat{\Gamma}$.*

Proof. Hodge theory implies that the torsion-free rank of the first homology of a compact Kähler manifold is even (see [6], p. 7), and this is the first betti number of its fundamental group Γ . A finite-sheeted covering of a compact Kähler manifold is again a compact Kähler manifold, so the first betti number of any subgroup of finite index in Γ is also even. By the preceding lemma, this completes the proof. ■

4.3. GOODNESS AND SURFACE SUBGROUPS. Goodness was defined in Section 2.4. By taking $\Gamma = S$ in the following proposition, we obtain a further proof that a surface group cannot have the same profinite completion as a free group. We shall then extend this result to (conjecturally all) residually free groups.

A group Γ is termed a **Poincaré duality group** over a field \mathbf{F} (or $\text{PD}_d(\mathbf{F})$ for short) if there is an integer d such that $H^*(\Gamma; \mathbf{F}) = H_{*-d}(\Gamma; \mathbf{F})$. The prototypes

for such groups are the fundamental groups of closed aspherical manifolds. We will write \mathbf{F}_p for the field with p elements.

PROPOSITION 4.9: *Let Γ be a group that contains a subgroup S that is $\text{PD}_d(\mathbf{F}_p)$ for some $d > 1$ and some prime p . If S is good and $\widehat{S} \rightarrow \widehat{\Gamma}$ is injective, then there does not exist a free group L such that $\widehat{\Gamma} \cong \widehat{L}$.*

Proof. Since free groups are good, $H^q(\widehat{L}; \mathbf{F}_p) = 0$ for all $q \geq 2$. Hence if $\widehat{\Gamma}$ were isomorphic to \widehat{L} , then from Corollary 2.17 we would have $H^2(H; \mathbf{F}_p) = 0$ for all $q \geq 2$ and all good subgroups $H < \Gamma$ with $\widehat{H} \rightarrow \widehat{\Gamma}$ injective. Now S is such a group, but $H^d(S; \mathbf{F}_p) = H_0(S; \mathbf{F}_p) = \mathbf{F}_p$. ■

As noted earlier, it was established in [23] that surface groups are good. The following lemma includes the case where Γ itself is a surface group.

COROLLARY 4.10: *If Γ contains a surface group S , and $\widehat{S} \rightarrow \widehat{\Gamma}$ is injective, then $\widehat{\Gamma}$ is not isomorphic to the profinite completion of any free group.*

A similar argument shows the following:

LEMMA 4.11: *If L is a non-abelian free group, then \widehat{L} does not contain the profinite completion of any surface group, nor that of any free abelian group of rank greater than 1.*

Remark 4.12: Note that \widehat{L} does contain surface subgroups S of arbitrary large genus (as shown in [10] for example) and free abelian subgroups of arbitrary rank, but the natural map $\widehat{S} \rightarrow \widehat{L}$ is never injective. The surface subgroup examples of [10] are in fact dense in \widehat{L} .

4.4. WORD HYPERBOLIC GROUPS. Next we single out a particular case of an application of Proposition 4.9 that is worth recording. This result connects to two well-known open problems about word hyperbolic groups, namely:

- (A) Does every 1-ended word-hyperbolic group contain a surface subgroup?
- (B) Is every word-hyperbolic group residually finite?

The first question, due to Gromov, was motivated by the case of hyperbolic 3-manifolds, and in this special case the question was settled recently by Kahn and Markovic [27]. Indeed, given [27], a natural strengthening of (A) above is to ask:

(A') Does every 1-ended word-hyperbolic group contain a quasi-convex surface subgroup?

THEOREM 4.13: *Suppose that every 1-ended word-hyperbolic group is residually finite and contains a quasi-convex surface subgroup. Then there exist no 1-ended word-hyperbolic group Γ and free group F such that $\widehat{\Gamma} \cong \widehat{F}$.*

Proof. Assume the contrary, and let Γ be a counter-example, with $\widehat{\Gamma} \cong \widehat{F}$ for some free group F . Let H be a quasi-convex surface subgroup of Γ . Note that the finite-index subgroups of H are also quasi-convex in Γ . Under the assumption that all 1-ended hyperbolic groups are residually finite, it is proved in [2] that H and all its subgroups of finite index must be separable in Γ . Hence by Lemma 2.7, the natural map $\widehat{H} \rightarrow \overline{H} < \widehat{\Gamma}$ is an isomorphism. But now Proposition 4.9 yields a contradiction. ■

COROLLARY 4.14: *Suppose that there exists a 1-ended word hyperbolic group Γ with $\widehat{\Gamma} \cong \widehat{F}$ for some free group F . Then either there exists a word-hyperbolic group that is not residually finite, or there exists a word-hyperbolic group that does not contain a quasi-convex surface subgroup.*

4.5. RESIDUALLY FREE GROUPS. A group Γ is called **residually free** if for every non-trivial element $g \in \Gamma$ there is a homomorphism ϕ_g from Γ to a free group such that $\phi_g(g) \neq 1$, and Γ is **fully residually free** if for every finite subset $X \subseteq \Gamma$ there is a homomorphism from Γ to a free group that restricts to an injection on X .

Following Sela [51], we use the term **limit group** to describe a finitely-generated group Γ that is fully residually free. This defines a rich class of groups, the prototypes for which are the free abelian groups and torsion-free Fuchsian groups (with the exception of cocompact groups of Euler characteristic -1). Indeed an arbitrary limit group can be built from these basic examples in a simple hierarchical manner, by starting with a join of n -tori, closed surfaces and compact graphs, and at each level attaching a torus along a coordinate circle to the space at the previous level, or attaching a hyperbolic surface along a boundary curve; one requires that the new space retracts onto the previous level. The finitely-generated subgroups of the fundamental groups of these spaces are precisely the limit groups; see [51, 28].

LEMMA 4.15: *If Γ is finitely-generated and residually free, then either Γ is a hyperbolic limit group, or else Γ has a free abelian subgroup of rank 2.*

Proof. Every finitely-generated residually free group Γ is a subgroup of a direct product of finitely many limit groups [8]. By projecting away from factors that it intersects trivially, we may assume that Γ intersects each of the factors. Limit groups are torsion-free, so if Γ intersects two factors then it contains a copy of \mathbb{Z}^2 . Otherwise, it is a subgroup of a limit group. Finitely-generated subgroups of limit groups are themselves limit groups, and non-hyperbolic limit groups contain \mathbb{Z}^2 . ■

Wilton [55] proved that if Γ is a limit group and H is a finitely-generated subgroup of Γ , then there is a subgroup $\Gamma_0 < \Gamma$ of finite index that contains H and retracts onto it. If one assumes only that Γ is finitely-generated and residually free, then the analogous statement is false in general. However, it is shown in [13] that, in this generality, Γ does virtually retract onto any subgroup that has a finite classifying space. Thus we obtain the following:

THEOREM 4.16: *If Γ is finitely-generated and residually free, then either Γ has a subgroup of finite index that retracts onto a surface group (of genus $g \geq 1$) or else Γ is a hyperbolic limit group that does not contain a surface subgroup.*

THEOREM 4.17: *Let Γ be a residually-free group and let F be a free group. If $\widehat{\Gamma} \cong \widehat{F}$, then either $\Gamma \cong F$ or else Γ is a hyperbolic limit group that does not contain a surface subgroup.*

Proof. It is easy to see that if $S < \Gamma$ is a virtual retract then the natural map $\widehat{S} \rightarrow \overline{S} < \widehat{\Gamma}$ is an isomorphism onto its image, so in the light of Theorem 4.16 we can apply Proposition 4.9. ■

Remark 4.18: It is believed by many that the second possibility stated in the above theorem does not arise, but this remains unknown for the moment. Using the hierarchical structure of limit groups described above, one sees that the crucial case to consider is one where Γ is the fundamental group of a graph of free groups with cyclic edge groups. There has been much recent work on the existence of surface subgroups in such groups; see [14], [22], [30], and [29] for example.

Remark 4.19: In the preceding proof, we exploited the fact that free groups and surface groups are good. The direct product of two good groups is again good (see [23] Proposition 3.4), so $F_k \times F_k$ is good. Now consider the fibre product $P = \{(x, y) \mid p(x) = p(y)\} < F_k \times F_k$ associated with an epimorphism $p : F_k \rightarrow Q$, where Q is a finitely-presented group with $H_2(Q, \mathbb{Z}) = 0$ that has no non-trivial finite quotients. The inclusion $u : P \rightarrow \Gamma := F_k \times F_k$ induces an isomorphism $\widehat{u} : \widehat{P} \rightarrow \widehat{\Gamma}$ (see [11]). Taking coefficients in a finite module M with trivial action, we find that $H^2(\widehat{\Gamma}, M)$ will be finitely-generated, since Γ is good. But a simple spectral sequence calculation shows that in many cases $H^2(P, M)$ is *not* finitely-generated (cf. [7]), and so P is *not* good. Thus we have an example of a finitely-generated residually free group that is *not* good. In contrast, all limit groups *are* good; see [23].

It follows that goodness is definitely a property of Γ and not of $\widehat{\Gamma}$: we have $P < \Gamma$ with $\widehat{P} \cong \widehat{\Gamma}$, but Γ is good while P is not.

5. Torsion in the profinite completions of Fuchsian groups

In this section we will consider the case of a non-elementary Fuchsian group Γ . In the cocompact case, any such group Γ has a presentation of the form:

$$\langle a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_r \mid x_i^{m_i} = 1 \ (i = 1, \dots, r), \prod_1^r x_i \prod_1^g [a_k, b_k] = 1 \rangle.$$

The parameters appearing in the presentation $(g; m_1, \dots, m_r)$ constitute the **signature** of Γ . The integers m_i are the **periods** of Γ , and g is its **genus**. Groups with signature $(0; p, q, r)$ are generally referred to as **(p, q, r) -triangle groups**. If the Fuchsian group is not cocompact, then it is a free product of cyclic groups.

We write $\mathfrak{cf}(\Gamma)$ to denote the set of conjugacy classes of maximal finite subgroups of a group Γ . Recall that a group Γ is called **conjugacy separable** if it has the property that if the images of two elements of Γ are conjugate in every finite quotient of G (equivalently, are conjugate in $\widehat{\Gamma}$), then those two elements are conjugate in Γ .

THEOREM 5.1: *If Γ is a finitely-generated Fuchsian group, then the natural inclusion $\Gamma \rightarrow \widehat{\Gamma}$ induces a bijection $\mathfrak{cf}(\Gamma) \rightarrow \mathfrak{cf}(\widehat{\Gamma})$. More precisely, every finite subgroup of $\widehat{\Gamma}$ is conjugate to a subgroup of Γ , and if two maximal finite subgroups of Γ are conjugate in $\widehat{\Gamma}$ then they are already conjugate in Γ .*

Proof. The finite subgroups of Fuchsian groups are cyclic, so the injectivity of $\text{cf}(\Gamma) \rightarrow \text{cf}(\widehat{\Gamma})$ is a special case of the fact that finitely-generated Fuchsian groups are conjugacy separable [53].

In order to prove surjectivity, we first suppose that Γ is not a triangle group. In this case, Γ splits as an amalgamated free product $\Gamma = A *_C B$, where the non-trivial groups A and B are free products of cyclic groups. By Bass–Serre theory, the finite free factors of A and B are in bijection with the conjugacy classes of maximal finite subgroups in Γ . Now by Corollary 2.20, we have $\widehat{\Gamma} \cong \widehat{A} \amalg_{\widehat{C}} \widehat{B}$ and by Theorem 2.18, every finite subgroup of $\widehat{\Gamma}$ is conjugate to a subgroup of \widehat{A} or \widehat{B} . Repeating this argument with A and B in place of Γ , we deduce that every finite subgroup of $\widehat{\Gamma}$ is conjugate to a subgroup of the profinite completion of one of the free factors of A or B . But each free factor Z_i is cyclic, so \widehat{Z}_i is torsion-free ($\cong \widehat{\mathbb{Z}}$) if Z_i is infinite, while $\widehat{Z}_i = Z_i$ if Z_i is finite. Thus every maximal finite subgroup of $\widehat{\Gamma}$ is conjugate to one of the Z_i .

It remains to prove surjectivity in the case where Γ is a hyperbolic triangle group, say the (p, q, r) -triangle group. Let H be a maximal finite subgroup of $\widehat{\Gamma}$. Since $\widehat{\Gamma}$ is residually finite, we can pass to a subgroup of arbitrarily large index that contains H ; in particular, we can make the index greater than pqr . Let $\Lambda \subset \Gamma$ be the intersection of Γ with this subgroup. Then the index of Λ in Γ is greater than pqr , so the rational Euler characteristic of Λ is less than -2 , which means that Λ is not a hyperbolic triangle group. Hence our previous argument applies, and H is conjugate in $\widehat{\Lambda}$ to a finite subgroup H' of Λ . Since H is maximal in $\widehat{\Gamma}$, it follows that this H' is maximal in Γ . ■

We can now use Theorem 5.1 to eliminate the latter case of Theorem 3.6.

COROLLARY 5.2: *Let Γ be a finitely generated Fuchsian group. Then $\widehat{\Gamma}$ does not contain a non-trivial finite normal subgroup.*

Proof. Suppose N is a non-trivial finite normal subgroup of $\widehat{\Gamma}$. By Theorem 5.1, since every finite subgroup of $\widehat{\Gamma}$ is conjugate to a subgroup of Γ , N is a subgroup of Γ . However, non-elementary Fuchsian groups do not contain finite, non-trivial normal subgroups. ■

Also, note that Theorem 5.1 provides another proof of the following special case of Corollary 2.16.

COROLLARY 5.3: *If Γ is a torsion-free Fuchsian group, then $\widehat{\Gamma}$ is torsion-free.*

6. The Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1, re-stated below for convenience.

THEOREM: *Let Γ_1 be a finitely-generated Fuchsian group and let Γ_2 be a lattice in a connected Lie group. If $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$, then $\Gamma_1 \cong \Gamma_2$.*

Proof of Theorem 1.1. By Theorem 3.6 and Corollary 5.2, we can now assume that both Γ_1 and Γ_2 are Fuchsian.

Now suppose that Γ_1 and Γ_2 are torsion-free. In this case, each of Γ_1 and Γ_2 is either a free group or a surface group — that is, isomorphic to the fundamental group of a closed orientable surface of genus at least 2. We proved (three times) in Section 4 that if $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ then both of Γ_1 and Γ_2 are free groups, or both are surface groups. But the isomorphism type of a free group or a surface group Γ is determined by the rank of its abelianisation, which by Corollary 2.11 is an invariant of $\widehat{\Gamma}$. Hence the theorem holds in this case.

Next, we note that it cannot be true that Γ_1 is cocompact and Γ_2 is not. For if this were the case, then we could pass to torsion-free subgroups of common finite index that would still have isomorphic profinite completions (see Corollary 2.9), which is impossible by the previous argument.

If neither Γ_1 nor Γ_2 is cocompact, then each is a free product of cyclic groups. We know that $b_1(\Gamma_1) = b_1(\Gamma_2)$, by Corollary 2.11, and so the number of infinite cyclic factors in each product is the same. And by Theorem 5.1, the finite cyclic factors, being in bijection with the conjugacy classes of maximal finite subgroups, are also the same. Hence the theorem holds in this case too.

It only remains to consider the case where both Γ_1 and Γ_2 are cocompact groups with torsion (by Corollary 5.3). The genus of Γ_i is determined by $b_1(\Gamma_i)$, so Corollary 2.11 tells us that Γ_1 and Γ_2 are of the same genus. The periods of Γ_i are the orders of representatives of the conjugacy classes of maximal finite subgroups of Γ_i , and so by Proposition 5.1 these must also be the same for Γ_1 and Γ_2 . Thus Γ_1 and Γ_2 have the same signature, and are therefore isomorphic.

This completes the proof of Theorem 1.1. ■

Remark 6.1: It is interesting to compare Theorem 1.1 with some of the findings in [24]. Among other things, it is shown in [24, Theorem 4.1] that if $\Gamma = A*B$ is a free product of finite groups A and B , then among all finitely-generated virtually free groups, Γ is determined by its profinite completion. Our Theorem 1.1 covers

free products of finite cyclic groups with arbitrarily many factors. On the other hand, examples are provided in [24] of finitely-generated virtually free groups that are not determined by their profinite completions.

7. Distinguishing non-uniform arithmetic Kleinian groups

A non-uniform arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is a discrete group commensurable (up to conjugacy) with a Bianchi group $\mathrm{PSL}(2, O_d)$, where O_d is the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{-d})$; see [38]. In this section we prove Theorem 1.3, re-stated below for convenience:

THEOREM: *Let Γ_1 be a non-uniform lattice in $\mathrm{PSL}(2, \mathbb{C})$, and let Γ_2 be a non-uniform irreducible arithmetic lattice in a semisimple Lie group G having trivial centre and no compact factors. If $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ then $G \cong \mathrm{PSL}(2, \mathbb{C})$. Moreover, if Γ_1 is arithmetic then the family of all Γ_2 with $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ divides into finitely many commensurability classes.*

After a series of reductions, we shall arrive at the case where Γ_2 is a torsion-free lattice in $\mathrm{SO}_0(n, 1)$ or $\mathrm{SU}(m, 1)$. The **peripheral subgroups** of Γ_2 are the maximal subgroups consisting entirely of parabolic elements; they are the fundamental groups of the cusps of the locally symmetric space with fundamental group Γ_2 . If $\Gamma_2 < \mathrm{SO}_0(n, 1)$ then these subgroups are virtually \mathbb{Z}^{n-1} , while if $\Gamma_2 < \mathrm{SU}(m, 1)$ they are commensurable with the integral Heisenberg group H_{2m-1} , which is a central extension of \mathbb{Z}^{2m-2} by \mathbb{Z} . We shall make use of the following:

LEMMA 7.1: *Let Γ be a non-uniform arithmetic lattice in $\mathrm{SO}_0(n, 1)$ or $\mathrm{SU}(m, 1)$ and let D be a peripheral subgroup of Γ . Then every subgroup of D is separable in Γ_2 , and D is good.*

Proof. The first part of the lemma is due to McReynolds; see [40, Theorem 1.3 and Corollaries 4.1 and 4.2]. For the second part, it suffices to show that the groups \mathbb{Z}^k and H_k are good, because goodness is preserved by commensurability; see [23, Lemma 10]. As noted above, both of these groups are extensions of a free abelian group by \mathbb{Z} , and so they are good by Lemma 2.12. ■

We will also use this theorem, which was proved in [16, Corollary 1.2]:

THEOREM 7.2: *Let N be any positive integer. There are only finitely many commensurability classes of non-uniform arithmetic lattices in $\mathrm{PSL}(2, \mathbb{C})$ that contain a lattice Γ with the property that \mathbf{H}^3/Γ has at most N cusps.*

Proof of Theorem 1.3. As before, Theorem 2.5 implies that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. First note that if necessary we can replace Γ_1 and Γ_2 by subgroups of finite index so as to assume that they are torsion-free. Indeed, if we first pass to a finite index subgroup of Γ_1 (which we will continue to refer to as Γ_1) to arrange that Γ_1 is torsion-free, then Corollary 2.9 will provide us with a subgroup of finite index in Γ_2 (which again we will continue to refer to as Γ_2) so that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. At this point, if Γ_2 is not torsion-free, then we can pass to a further subgroup of finite index in Γ_2 that is torsion-free, and use Corollary 2.9 again to pass to the corresponding subgroup of Γ_1 .

A standard Alexander–Lefschetz duality argument implies that $b_1(\Gamma_1)$ is positive, so $b_1(\Gamma_2)$ is positive too, by Corollary 2.11. Given this, it follows that Γ_2 cannot be a lattice in a group of real rank at least 2, because the Margulis normal subgroup theorem [39] implies that such groups have finite abelianisation (note that this is where irreducibility is invoked). Similarly, Γ_2 cannot be a lattice in a rank 1 Lie group with property (T), and this rules out $\mathrm{Sp}(n, 1)$ ($n \geq 2$) and the isometry group of the Cayley hyperbolic plane. The only remaining possibilities (up to finite index) are $(\mathrm{P})\mathrm{SO}_0(n, 1)$ and $(\mathrm{P})\mathrm{SU}(n, 1)$.

We dealt with $\mathrm{SO}_0(2, 1) = \mathrm{PSL}(2, \mathbb{R})$ in Theorem 1.1. Hence it remains to rule out $\mathrm{SO}_0(n, 1)$ for $n \geq 4$ and $\mathrm{SU}(m, 1)$ for $m \geq 2$. In these cases, as discussed above, we have a peripheral subgroup $D < \Gamma_2$ that is good, with the property that the closure of its image in $\widehat{\Gamma}_2$ is isomorphic to \widehat{D} . But $D = \mathbb{Z}^{n-1}$ or H_{2m-1} is the fundamental group of a closed, orientable manifold, of dimension $n - 1 \geq 3$ in the former case, and dimension $2m - 1 \geq 3$ in the latter. As such, D satisfies Poincaré duality with coefficients in an arbitrary finite field. It therefore follows from Corollary 2.17 that $\widehat{\Gamma}_2$ cannot be isomorphic to the profinite completion of a good group of cohomological dimension less than 3. On the other hand, Γ_1 is such a group: it has cohomological dimension 2 because it is the fundamental group of an aspherical 3-manifold with non-empty boundary, and it is good by Theorem 2.13. At this point, we have proved that Γ_2 is a lattice in $\mathrm{SO}_0(3, 1) \cong \mathrm{PSL}(2, \mathbb{C})$.

To complete the proof of Theorem 1.3, we suppose now that Γ_1 and Γ_2 are non-uniform arithmetic lattices in $\mathrm{PSL}(2, \mathbb{C})$ and that $b_1(\Gamma_1) = n$. Then

$b_1(\Gamma_2) = n$ by Corollary 2.11, and so by a standard fact about 3-manifold groups with torus boundary components, the number of cusps of \mathbf{H}^3/Γ_2 is at most n . Application of Theorem 7.2 now completes the proof of Theorem 1.3. ■

If we drop the assumption of irreducibility of the lattice Γ_2 in Theorem 1.3, then some observations can be made, but at this stage we are unable to reach the same conclusion. For instance, at present we are unable to answer the following question.

Question 7.3: Let Γ_1 be a non-uniform lattice in $\text{PSL}(2, \mathbb{C})$, and let $\Gamma_2 = F \times \Delta$ where F is a free group of rank $r > 1$ and Δ is a cocompact lattice in $\text{SU}(n, 1)$. Is it possible that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$?

8. Finite quotients of triangle groups

In this section we give a more direct proof of the fact that triangle groups are distinguished among themselves by their finite quotients, and give some explicit quotients that distinguish non-triangle groups from triangle groups.

THEOREM 8.1: *If Γ and Σ are triangle groups for which $\mathcal{C}(\Gamma) = \mathcal{C}(\Sigma)$, then $\Gamma \cong \Sigma$.*

We will use the notation $\Delta(r, s, t) = \langle x, y, z \mid xyz = x^r = y^s = z^t = 1 \rangle$ for the (r, s, t) triangle group. Each triangle group is called **spherical**, **Euclidean** or **hyperbolic** according to whether the quantity $1/r + 1/s + 1/t$ is greater than, equal to or less than 1, respectively. Note that $\Delta(r, s, t)$ is isomorphic to $\Delta(u, v, w)$ whenever the triple (u, v, w) is a permutation of the triple (r, s, t) , and hence up to isomorphism we may assume that $r \leq s \leq t$.

The spherical triangle groups are $\Delta(1, n, n)$, $\Delta(2, 2, n)$, $\Delta(2, 3, 3)$, $\Delta(2, 3, 4)$ and $\Delta(2, 3, 5)$, which are isomorphic to C_n (cyclic), D_n (dihedral of order $2n$), A_4 , S_4 and A_5 , respectively. The Euclidean triangle groups are $\Delta(2, 3, 6)$, $\Delta(2, 4, 4)$ and $\Delta(3, 3, 3)$, each of which is an extension of a free abelian group of rank 2 by a cyclic group C_t (with $t = 6, 4$ and 3 , respectively). In particular, the spherical triangle groups are finite, while the Euclidean triangle groups are infinite but soluble. In contrast, all hyperbolic triangle groups are infinite but insoluble. See [18] for further details.

The latter categorisation makes the spherical and Euclidean triangle groups easy to distinguish from others by their finite quotients, and so we will restrict

our attention to the hyperbolic ones, which are the Fuchsian groups with signature $(0; r, s, t)$ where $1/r + 1/s + 1/t < 1$. The most famous of these is $\Delta(2, 3, 7)$, as it gives the largest value of $1/r + 1/s + 1/t$ less than 1, and its non-trivial quotients are the Hurwitz groups (see [17]).

We will define a finite group G to be (r, s, t) -**generated** if G can be generated by elements a, b and c of (precise) orders r, s and t such that $abc = 1$. For any hyperbolic triple (r, s, t) , the set of (r, s, t) -generated groups is non-empty, by residual finiteness of $\Delta(r, s, t)$, but in most cases $\Delta(r, s, t)$ can also have ‘non-smooth’ quotients, in which the orders of the generators are not preserved.

The following theorem is a direct consequence of observations made by Macbeath [37] on (r, s, t) -generation of the groups $\text{PSL}(2, q)$, and will be critical to our proof of Theorem 8.1:

THEOREM 8.2: *Let (r, s, t) be any hyperbolic triple other than $(2, 5, 5)$, $(3, 4, 4)$, $(3, 3, 5)$, $(3, 5, 5)$ or $(5, 5, 5)$, and let p be any prime. If p^f is the smallest power of p for which $\text{PSL}(2, p^f)$ contains elements of orders r, s and t , then either $\text{PSL}(2, p^f)$ is (r, s, t) -generated, or f is even and $\text{PGL}(2, p^{f/2})$ is (r, s, t) -generated. In particular, $\text{PSL}(2, p)$ itself is (r, s, t) -generated whenever it contains elements of orders r, s and t .*

The triples $(2, 5, 5)$, $(3, 4, 4)$, $(3, 3, 5)$, $(3, 5, 5)$ and $(5, 5, 5)$, together with the spherical triples and the triple $(3, 3, 3)$, were called **exceptional** by Macbeath. Note that the group $A_5 \cong \text{PSL}(2, 5)$ is $(2, 5, 5)$ -, $(3, 3, 5)$ -, $(3, 5, 5)$ - and $(5, 5, 5)$ -generated, while the group S_4 is $(3, 4, 4)$ -generated.

We will also make use of the fact that if the finite group G is (r, s, t) -generated, then G is a group of conformal automorphisms of a compact Riemann surface S of genus g , where

$$2 - 2g = |G| \left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1 \right)$$

as a consequence of the Riemann–Hurwitz formula. The kernel K of the corresponding smooth homomorphism from $\Delta(r, s, t)$ onto G is the fundamental group of S , and is itself a Fuchsian group, with signature $(2g; -)$. In particular, K is generated by $2g$ elements $a_1, b_1, \dots, a_g, b_g$ subject to a single defining relation $[a_1, b_1] \dots [a_g, b_g] = 1$. Now for any positive integer n , the subgroup $K'K^{(n)}$ generated by the derived subgroup K' and the n th powers of all elements of K is characteristic in K and hence normal in $\Delta(r, s, t)$, and the quotient

$\Delta(r, s, t)/K'K^{(n)}$ is then isomorphic to an extension by G of an abelian subgroup $K/K'K^{(n)}$ of rank $2g$ and exponent n (and order n^{2g}). Thus for any such G , we can construct an infinite family of smooth quotients of $\Delta(r, s, t)$, to help distinguish $\Delta(r, s, t)$ from other triangle groups.

We can now proceed to prove Theorem 8.1. To do that, we will suppose that $\Gamma = \Delta(r, s, t)$ and $\Sigma = \Delta(u, v, w)$, where (r, s, t) and (u, v, w) are hyperbolic triples with $r \leq s \leq t$ and $u \leq v \leq w$, and that $\mathcal{C}(\Gamma) = \mathcal{C}(\Sigma)$. We will prove in steps that $(r, s, t) = (u, v, w)$.

LEMMA 8.3:

- (a) $\gcd(r, s, t) = \gcd(u, v, w)$, and (b) $\frac{rst}{\text{lcm}(r, s, t)} = \frac{uvw}{\text{lcm}(u, v, w)}$, and also
- (c) $\text{lcm}(\gcd(r, s), \gcd(r, t), \gcd(s, t)) = \text{lcm}(\gcd(u, v), \gcd(u, w), \gcd(v, w))$.

Proof. The abelianisation of $\Gamma = \Delta(r, s, t)$ is $C_d \times C_e$, where $d = \gcd(r, s, t)$ and $de = rst/\text{lcm}(r, s, t)$ and $e = \text{lcm}(\gcd(r, s), \gcd(r, t), \gcd(s, t))$, and similarly for $\Sigma = \Delta(u, v, w)$. Since Γ and Σ have the same finite abelian quotients, the results follow. ■

LEMMA 8.4: Γ and Σ have the same set of (r, s, t) -generated quotients, and the same set of (u, v, w) -generated quotients. Moreover, $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}$.

Proof. The first two assertions are easy, since Γ and Σ have exactly the same finite quotients. For the last part, without loss of generality we may suppose $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} \leq \frac{1}{u} + \frac{1}{v} + \frac{1}{w}$. Now let G be any (r, s, t) -generated quotient of Γ . Then G is also a finite quotient of Σ . Next let u', v' and w' be divisors of u, v and w (respectively) such that G is (u', v', w') -generated, and such that $\frac{1}{u'} + \frac{1}{v'} + \frac{1}{w'}$ is as small as possible subject to those conditions. Then in particular, $\frac{1}{u'} + \frac{1}{v'} + \frac{1}{w'} \geq \frac{1}{u} + \frac{1}{v} + \frac{1}{w} \geq \frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.

For any n coprime to $|G|$, the largest quotient of Γ that is an extension of an abelian group of exponent n by G has order $n^{2g}|G|$, where $2 - 2g = |G|(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1)$. On the other hand, the largest quotient of Σ that is an extension of an abelian group of exponent n by G must be a smooth quotient of the (u', v', w') triangle group and hence has order $n^{2g'}|G|$, where $2 - 2g' = |G|(\frac{1}{u'} + \frac{1}{v'} + \frac{1}{w'} - 1)$.

Since Γ and Σ have the same quotients, it follows that $g' = g$, and so $\frac{1}{u'} + \frac{1}{v'} + \frac{1}{w'} = \frac{1}{r} + \frac{1}{s} + \frac{1}{t}$. The inequality obtained at the end of the first paragraph now gives $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = \frac{1}{r} + \frac{1}{s} + \frac{1}{t}$. It also gives $\frac{1}{u'} + \frac{1}{v'} + \frac{1}{w'} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}$, so $(u', v', w') = (u, v, w)$, and hence G is (u, v, w) -generated.

In particular, every (r, s, t) -generated finite group is also (u, v, w) -generated. The converse holds by the same argument, with the roles of (r, s, t) and (u, v, w) reversed. ■

LEMMA 8.5: $rst = uvw$, and $\text{lcm}(r, s, t) = \text{lcm}(u, v, w)$, and $rs + rt + st = uv + uw + vw$.

Proof. For any prime divisor p of rst , let p^α, p^β and p^γ be the largest powers of p dividing r, s and t , ordered in such a way that $\alpha \leq \beta \leq \gamma$. Then p^α must be the largest power of p dividing $\text{gcd}(r, s, t)$, while p^β is the largest power of p dividing $\text{lcm}(\text{gcd}(r, s), \text{gcd}(r, t), \text{gcd}(s, t))$, and p^γ is the largest power of p dividing $\text{lcm}(r, s, t)$. Also $p^{\alpha+\beta+\gamma}$ is the largest power of p dividing rst , and so $p^{\alpha+\beta}$ is the largest power of p dividing $\frac{rst}{\text{lcm}(r, s, t)}$. Furthermore, either $\beta = \gamma$, or p^γ is the largest power of p dividing the denominator of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{rs+rt+st}{rst}$ when the latter is expressed in reduced form. (To see the last part, note that $rs + rt + st$ is divisible by $p^{\alpha+\beta}$ but not $p^{\alpha+\beta+1}$ when $\beta < \gamma$.)

Hence the largest powers of p dividing r, s and t are determined by the quantities $\text{gcd}(r, s, t)$, $\frac{rst}{\text{lcm}(r, s, t)}$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$. By Lemmas 8.3 and 8.4, these three quantities are the same for the triple (u, v, w) , and hence the largest powers of p dividing u, v and w are equal to those for r, s and t , in some order. As this holds for every prime p , the stated equalities follow easily. ■

We now deal with many special cases of Theorem 8.1.

PROPOSITION 8.6: *The conclusion of Theorem 8.1 holds whenever*

- (a) (r, s, t) is one of the exceptional triples $(2, 5, 5), (3, 4, 4), (3, 3, 5), (3, 5, 5)$ or $(5, 5, 5)$, or
- (b) the triples (r, s, t) and (u, v, w) have an entry in common, or
- (c) two or more of r, s and t are even.

Proof. (a) This follows from Lemma 8.5, since $(2, 5, 5)$ is the only hyperbolic triple with $rst = 50$, and $(3, 4, 4)$ is the only hyperbolic triple with $rst = 48$, $\text{lcm}(r, s, t) = 12$ and $\text{gcd}(r, s, t) = 1$, and $(3, 3, 5)$ is the only hyperbolic triple with $rst = 45$, and $(3, 5, 5)$ is the only hyperbolic triple with $rst = 75$, and finally, $(5, 5, 5)$ is the only hyperbolic triple with $rst = 125$.

(b) Suppose for example that $t = w$. Then $rs = \frac{rst}{t} = \frac{uvw}{w} = uv$, and then since $rs + (r + s)t = rs + rt + st = rst(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}) = uvw(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}) = uv + uw + vw = uv + (u + v)w$, we find that $r + s = u + v$. It is now an

easy exercise to deduce from $rs = uv$ and $r + s = u + v$ that $\{r, s\} = \{u, v\}$, and hence $(r, s, t) = (u, v, w)$. The same argument works for all other possible coincidences between entries of (r, s, t) and (u, v, w) .

(c) Suppose two or more of r, s and t are even. Then $\text{lcm}(\text{gcd}(r, s), \text{gcd}(r, t), \text{gcd}(s, t))$ is even, and therefore so is $\text{lcm}(\text{gcd}(u, v), \text{gcd}(u, w), \text{gcd}(v, w))$, and hence two or more of u, v and w are even. Also if all three of r, s and t are even, then $\text{gcd}(r, s, t)$ is even, hence so is $\text{gcd}(u, v, w)$, and therefore all three of u, v and w are even. Now let $m = \max(t, w)$ if all three of r, s and t are even, or otherwise let m be the largest odd integer among r, s, t, u, v and w . Then the dihedral group D_m of order $2m$, which is $(2, 2, m)$ -generated, is a quotient of Γ or Σ , and hence must also be a quotient of the other. By definition of m , it follows that m appears in both triples (r, s, t) and (u, v, w) , and hence by part (b), we have $(r, s, t) = (u, v, w)$. ■

To continue with the proof, we require some information about the groups $\text{PSL}(2, p)$, for p prime. When p is odd, the orders of the elements of $\text{PSL}(2, p)$ are precisely the divisors of $p, \frac{p-1}{2}$ and $\frac{p+1}{2}$ (see [54, Chapter 3.6] for example). Note that the integers $p, \frac{p-1}{2}$ and $\frac{p+1}{2}$ are pairwise coprime, so the order of any non-trivial element of $\text{PSL}(2, p)$ divides exactly one of them.

Now define the L_2 -set of a triple (k, l, m) to be the (unique) set of pairwise coprime positive integers whose least common multiple is the same as that of $\{k, l, m\}$ and which has the property that each of k, l and m divides exactly one member of that set. For example, if k, l and m are themselves pairwise coprime, then the L_2 -set of the triple (k, l, m) is just $\{k, l, m\}$, while if $\text{gcd}(k, lm) = 1$ but $\text{gcd}(l, m) > 1$ then its L_2 -set is $\{k, \text{lcm}(l, m)\}$, and if $\text{gcd}(k, l) > 1$ and $\text{gcd}(l, m) > 1$ then its L_2 -set is $\{\text{lcm}(k, l, m)\}$. Note that every maximal prime-power divisor of $\text{lcm}(k, l, m)$ divides exactly one member of the L_2 -set.

It follows from Macbeath's theorem (Theorem 8.2) that if the triple (k, l, m) is non-exceptional, then the group $\text{PSL}(2, p)$ is (k, l, m) -generated if and only if each member of the L_2 -set of the triple (k, l, m) is equal to p or a divisor of $\frac{p \pm 1}{2}$. This enables us to prove the following:

LEMMA 8.7: *If the triples (r, s, t) and (u, v, w) are non-exceptional, then they have the same L_2 -set.*

Proof. Suppose that the L_2 -sets of (r, s, t) and (u, v, w) are distinct. Then, by swapping (r, s, t) and (u, v, w) if necessary, we see that there must exist maximal prime-power divisors q_1 and q_2 of $\text{lcm}(r, s, t) = \text{lcm}(u, v, w)$ such that q_1q_2 divides one member of the L_2 -set of (u, v, w) , but q_1 and q_2 divide different members of the L_2 -set of (r, s, t) , say m_1 and m_2 . Now by the Chinese Remainder Theorem and Dirichlet's theorem on primes in arithmetic progression, there are infinitely many (odd) primes p such that $p \equiv 1 \pmod{2m_1}$ while $p \equiv -1 \pmod{2m_2}$ and $p \equiv -1 \pmod{2n}$ for any other member n of the L_2 -set of (r, s, t) . For any such prime p , the group $\text{PSL}(2, p)$ is (r, s, t) -generated, since $p \equiv \pm 1 \pmod{2m}$ for every member m of the L_2 -set of (r, s, t) . On the other hand, $\text{PSL}(2, p)$ has no element of order q_1q_2 , since q_1 divides $\frac{p-1}{2}$ while q_2 divides $\frac{p+1}{2}$, and therefore $\text{PSL}(2, p)$ cannot be (u, v, w) -generated, contradiction. ■

COROLLARY 8.8: *If one of r, s, t is coprime to each of the other two, then $(r, s, t) = (u, v, w)$. In particular, if r, s and t are pairwise coprime, or equivalently, if the (r, s, t) triangle group Γ is perfect, then $(r, s, t) = (u, v, w)$.*

Proof. Clearly we need only prove the first assertion, and then the rest follows. So suppose that $(r, s, t) \neq (u, v, w)$, and also, say, that $\text{gcd}(r, st) = 1$. (The other two cases are similar.) Now $uvw = rst$ by Lemma 8.5, and each of u, v and w is distinct from r, s and t , by Proposition 8.6(b). Hence at least one of u, v and w divides neither r nor st , and so must be of the form cd , where c and d are non-trivial divisors of r and st respectively. It follows that the L_2 -sets of (r, s, t) and (u, v, w) are distinct, contradiction. ■

The next step requires a further general observation about L_2 -sets.

LEMMA 8.9: *For every triple (k, l, m) such that $k, l, m > 1$ and at most one of k, l, m is even, and for every integer $q > 3$ that does not divide any of the members of the L_2 -set of (k, l, m) , there exists a finite quotient G of the (k, l, m) triangle group such that G has no element of order q .*

Proof. This is easy to see for the exceptional triples: we can take $G = A_4$ for $(k, l, m) = (2, 3, 3)$, or $G = S_4$ for $(k, l, m) = (2, 3, 4)$, or $G = A_5$ for $(k, l, m) = (2, 3, 5), (2, 5, 5), (3, 3, 5), (3, 5, 5)$ or $(5, 5, 5)$ (and also $G = C_3 \times C_3$ for $(k, l, m) = (3, 3, 3)$). For any non-exceptional triple (k, l, m) , we can take $G = \text{PSL}(2, p)$, where p is a prime such that $p \equiv \pm 1$ modulo twice each of the members of the L_2 -set of (k, l, m) , but $p \not\equiv \pm 1$ modulo $2q$. ■

COROLLARY 8.10: *The integers u, v and w do not have non-trivial divisors u', v' and w' such that one of r, s and t is coprime to each of $6, u', v'$ and w' .*

Proof. Suppose to the contrary that $q \in \{r, s, t\}$ is coprime to $6, u', v'$ and w' . Then there exists a finite quotient G of the triangle group $\Delta(u', v', w')$ such that G has no non-trivial element of order dividing q . But then this group G is a quotient of $\Sigma = \Delta(u, v, w)$ but not of $\Gamma = \Delta(r, s, t)$, contradiction. \blacksquare

(As an illustration, consider the triples $(13, 15, 117)$ and $(9, 39, 65)$, which satisfy the conclusions of Lemmas 8.3, 8.4 and 8.5. We can ‘suppress’ $q = 13$ by taking $(u', v', w') = (9, 3, 5)$, and then find that $\text{PSL}(2, 19)$ is a quotient of $\Delta(9, 39, 65)$ but not of $\Delta(13, 15, 117)$.)

The observations we have made so far are sufficient to distinguish most triangle groups from each other, using just abelian, dihedral and 2-dimensional projective quotients. But these are not completely sufficient. For example, consider the triples $(15, 42, 63)$ and $(21, 21, 90)$, which satisfy the conclusions of Lemmas 8.3, 8.4 and 8.5, but do not satisfy the hypothesis of Corollary 8.8 and do not admit the kinds of divisors met in Corollary 8.10. For such triples, we need to consider further types of quotients, and it turns out that direct products give us almost all we need to complete a proof of the theorem. We will use the easily proved fact that if G and H are finite groups that are (r_1, s_1, t_1) - and (r_2, s_2, t_2) -generated, say by element triples (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, and we let $r = \text{lcm}(r_1, r_2)$, $s = \text{lcm}(s_1, s_2)$ and $t = \text{lcm}(t_1, t_2)$, then some subgroup of the direct product $G \times H$ is (r, s, t) -generated, by the triple $((x_1, x_2), (y_1, y_2), (z_1, z_2))$.

LEMMA 8.11: *If q_1 and q_2 are coprime positive integers, each greater than 3, such that $q_1 q_2$ divides at least one of u, v and w , then either $q_1 q_2$ divides at least one of r, s and t , or otherwise one of r, s and t is prime and equal to q_1 or q_2 .*

Proof. Suppose $q_1 q_2$ divides at least one of u, v and w , but divides none of r, s and t . Choose non-trivial divisors r_1 and r_2 of r , and non-trivial divisors s_1 and s_2 of s , and non-trivial divisors t_1 and t_2 of t , as large as possible, such that

- (i) $r = \text{lcm}(r_1, r_2)$, $s = \text{lcm}(s_1, s_2)$, and $t = \text{lcm}(t_1, t_2)$,
- (ii) each of r_1, s_1 and t_1 is coprime to q_1 to q_2 , with at least one being coprime to q_1 and at least one being coprime to q_2 , and
- (iii) no member of the L_2 -set of (r_2, s_2, t_2) is divisible by q_1 or q_2 .

It is an easy exercise to see that such a choice is always possible. For example, write $\text{lcm}(r, s, t)$ as $m_1 m_2$, where $\text{gcd}(m_1, m_2) = 1$ and q_1 divides m_1 while q_2 divides m_2 . If q_1 divides r , then take $r_1 = \text{gcd}(r, m_1)$, and take r_2 as $\text{gcd}(r, m_2)$ if the latter is positive, or the largest divisor of r not divisible by q_1 if r divides m_1 . Alternatively, if q_2 divides r , then take $r_1 = \text{gcd}(r, m_2)$, and take r_2 as $\text{gcd}(r, m_1)$ if the latter is positive, or the largest divisor of r not divisible by q_2 if r divides m_2 . If neither q_1 nor q_2 divides r , then take r_1 and r_2 to be $\text{gcd}(r, m_1)$ and $\text{gcd}(r, m_2)$ (in either order) if these are both positive, or $r_1 = r_2 = r$ if r divides m_1 or m_2 . Then make the analogous choices for s_1 and s_2 , and similarly for t_1 and t_2 .

With r_i, s_i and t_i chosen this way, it is not difficult to see that (r_1, s_1, t_1) is non-exceptional, and more importantly, there exists some prime p for which $\text{PSL}(2, p)$ is (r_1, s_1, t_1) -generated but has no element of order $q_1 c$ with $\text{gcd}(c, q_2) > 1$ or order $q_2 d$ with $\text{gcd}(d, q_1) > 1$. (Hence in particular, $\text{PSL}(2, p)$ has no element of order $q_1 q_2$.)

Now if the triple (r_2, s_2, t_2) consists of integers greater than 1, let G be any finite group that is (r_2, s_2, t_2) -generated but has no element of order divisible by q_1 or q_2 . Then some subgroup of the direct product $\text{PSL}(2, p_1) \times G$ is (r, s, t) -generated, but has no element of order $q_1 q_2$, and hence cannot be (u, v, w) -generated, contradiction. Consequently, at least one of r_2, s_2 or t_2 is 1, and so (by our choice of r_i, s_i and t_i) at least one of r, s and t must be equal to q_1 or q_2 , and be prime. ■

We now have enough to prove the main theorem.

Proof of Theorem 8.1. Assume the theorem is false, so that $\Gamma = \Delta(r, s, t)$ and $\Sigma = \Delta(u, v, w)$ have the same finite quotients, but $(r, s, t) \neq (u, v, w)$.

Consider the largest of the six integers r, s, t, u, v and w . By swapping the roles of (r, s, t) and (u, v, w) if necessary, we may assume that $w = \max\{r, s, t, u, v, w\}$. Then by Proposition 8.6(b), we know that w is greater than each of r, s and t (and in particular, w cannot divide r, s or t).

If w is a prime-power, then w must divide $\text{lcm}(r, s, t) = \text{lcm}(u, v, w)$ and so divides at least one of r, s or t , contradiction. Thus w is composite, say $w = q_1 q_2$, with $\text{gcd}(q_1, q_2) = 1$ and $1 < q_1 < q_2 < w$. Moreover, $q_1 q_2 = w$ divides none of r, s and t . Hence by Lemma 8.11 and some elementary number theory, we find that one of the following must hold:

- (a) $q_1 = 2$ and q_2 is an odd prime-power,
- (b) $q_1 = 3$ and q_2 is a prime-power (not divisible by 3),
- (c) one of q_1 and q_2 is prime and equal to one (or more) of r, s and t , and the other is a prime-power, or
- (d) $w = 6p$ where p is a prime greater than 3, and p is equal to one (or more) of r, s and t .

We will eliminate each of these four cases in turn.

Case (a): Here we can write $q_2 = p^k$ where p is an odd prime. Then since p^k divides $\text{lcm}(u, v, w) = \text{lcm}(r, s, t)$, we know that p^k divides at least one of r, s and t , and then since $\max\{r, s, t\} < w = 2p^k$, it follows that p^k is equal to at least one of r, s and t . Similarly, another one of r, s and t is even. Next, by the argument used in the proof of Lemma 8.5, we can write $\{r, s, t\} = \{2bp^i, cp^j, p^k\}$ and $\{u, v, w\} = \{dp^i, ep^j, 2p^k\}$, where b, c, d and e are positive integers coprime to p . Note also that $bc = de$ since $rst = uvw$, and that $i < k$ and $j \leq k$ since $\max\{r, s, t\} < w = 2p^k$.

Now $rs + rt + st = 2bcp^{i+j} + 2bp^{i+k} + cp^{j+k}$ while $uv + uw + vw = dep^{i+j} + 2dp^{i+k} + 2ep^{j+k}$, and since these are equal by Lemma 8.5, we find that $2bc + 2bp^{k-j} + cp^{k-i} = de + 2dp^{k-j} + 2ep^{k-i}$. Hence if $j < k$ then p divides $2bc - de = 2bc - bc = bc$, which is impossible, and therefore $j = k$. In turn, this forces $c = 1$ (because $cp^k = cp^j \leq \max\{r, s, t\} < w = 2p^k$) and then $e = 2$ (because p^k cannot lie in both $\{r, s, t\}$ and $\{u, v, w\}$, by Proposition 8.6(b)), but that is impossible, because at most one of u, v, w is even (by Proposition 8.6(c)).

Case (b): As in case (a), we can write $q_2 = p^k$ where p is prime, and this time we deduce that at least one of r, s and t is equal to p^k or $2p^k$.

First, suppose that $p^k \in \{r, s, t\}$. Then $p \neq 2$, for otherwise the fact that at most one of r, s and t is even would imply that p^k is coprime to the other two entries of the triple (r, s, t) , which is impossible by Corollary 8.8. Thus $p \geq 5$. Next, as in case (a), we can write $\{r, s, t\} = \{3bp^i, cp^j, p^k\}$ and $\{u, v, w\} = \{dp^i, ep^j, 3p^k\}$, where b, c, d and e are positive integers coprime to p , with $bc = de$, and $i < k$ while $j \leq k$. Equating $rs + rt + st$ with $uv + uw + vw$ and then dividing by p^{i+j} gives $3bc + 3bp^{k-j} + cp^{k-i} = de + 3dp^{k-j} + 3ep^{k-i}$.

If $j < k$, then p divides $3bc - de = 2bc$, which is impossible, so $j = k$. This further implies that $c = 1$ or 2 and $e = 2$ or 3 and $c \neq e$, by Proposition 8.6(b) and the maximality of $w = 3p^k$. Also $d = 1$, or else we could divide through by powers of p and apply Corollary 8.10. Hence in particular, $bc = e$. Now if $c = 2$

then $e = 2 = c$, contradiction, so $c = 1$, and $b = e \in \{2, 3\}$. If $b = 2$, however, then comparison of $rs + rt + st$ with $uv + uw + vw$ gives $12 + p^{k-i} = 5 + 6p^{k-i}$ and so $5p^{k-i} = 7$, which is impossible, and on the other hand, if $b = 3$, then $\text{lcm}(r, s, t) = \text{lcm}(9p^i, p^k, p^k) = 9p^k$ while $\text{lcm}(u, v, w) = \text{lcm}(p^i, 3p^k, 3p^k) = 3p^k$, another contradiction.

Thus $p^k \notin \{r, s, t\}$, and it follows that one of r, s, t is equal to $2p^k$. This time we can write $\{r, s, t\} = \{3bp^i, cp^j, 2p^k\}$ and $\{u, v, w\} = \{dp^i, ep^j, 3p^k\}$, where b, c, d and e are positive integers coprime to p , with $2bc = de$, and again $i < k$ while $j \leq k$. Equating $rs + rt + st$ with $uv + uw + vw$ and dividing by p^{i+j} gives $3bc + 6bp^{k-j} + 2cp^{k-i} = de + 3dp^{k-j} + 3ep^{k-i}$, and hence if $j < k$ we find that p divides $3bc - de = 3bc - 2bc = bc$, contradiction. Thus $j = k$. Now $c \leq 2$ (by maximality of $w = 3p^k$), but on the other hand, $c \neq 1$ since $p^k \notin \{r, s, t\}$, and $c \neq 2$ since at most one of r, s, t can be even, so again we reach a contradiction.

Case (c): Here each q_i is greater than 3, for otherwise case (a) or (b) applies, and so each q_i is coprime to 6.

If q_i is prime and lies in $\{r, s, t\}$, then divide each of u, v, w by the highest power of q_i possible, to obtain a triple (u', v', w') of integers each of which is coprime to q_i . Then by Corollary 8.10, at least one of these integers must be 1, so at least one of u, v and w is a power of q_i . By Proposition 8.6(b), however, none of them can equal q_i , so at least one is divisible by q_i^2 . On the other hand, none of them is divisible by q_2^2 , since $q_2^2 > q_1q_2 = w = \max\{r, s, t, u, v, w\}$, and therefore $i = 1$. Thus q_1 is prime and lies in $\{r, s, t\}$, and at least one of u, v, w is divisible by q_1^2 .

Let q_1^γ be the largest power of q_1 that equals one or more of u, v, w . Then $\gamma > 1$, and q_1^γ divides one or more of r, s, t , since $\text{lcm}(r, s, t) = \text{lcm}(u, v, w)$. Similarly q_2 divides one or more of r, s, t , but none of r, s, t is divisible by $q_1^\gamma q_2$ since $q_1^\gamma q_2 \geq q_1q_2 = w$. Thus we can write $\{r, s, t\} = \{q_1, bq_1^\gamma, cq_2\}$ and $\{u, v, w\} = \{q_1^\gamma, d, q_1q_2\}$, where b, c and d are positive integers with b coprime to q_1 , and $1 < b < q_2$ and $c < q_1$ (by Proposition 8.6(b) and maximality of w), and $d = bc$ (since $rst = uvw$). In particular, each of b, c and d is coprime to q_1 .

Now if $b > 3$, then since $\gamma > 1$ we find that bq_1^γ divides none of $\{q_1^\gamma, d, q_1q_2\} = \{u, v, w\}$, and then Lemma 8.11 gives a contradiction. Hence $b = 2$ or 3. On the other hand, comparing $rs + rt + st$ with $uv + uw + vw$ gives $bq_1^\gamma + cq_2 + bcq_1^{\gamma-1}q_2 = dq_1^{\gamma-1} + q_1^\gamma q_2 + dq_2$, from which it follows that q_1 divides $dq_2 - cq_2 = bcq_2 - cq_2 = (b - 1)cq_2$, which is impossible (since $q_1 > 3 \geq b$).

Case (d): In this final case, we can divide each of u, v, w by the highest power of p possible, to obtain a triple (u', v', w') of integers each of which is coprime to p . By Corollary 8.10, at least one of u', v', w' must be 1, so at least one of u, v, w is a power of p . On the other hand, by Proposition 8.6(b), none of u, v, w can equal p , so at least one of u, v, w is divisible by p^2 . Therefore p^2 must divide at least one of r, s, t (since $\text{lcm}(r, s, t) = \text{lcm}(u, v, w)$), and again by Proposition 8.6(b), it follows that at least one of r, s, t, u, v, w is divisible by kp^2 for some $k \geq 2$. This, however, is impossible because $kp^3 \geq 2p^2 > 6p = w$. ■

We acknowledge the use of MAGMA [9] in helping us find a way to this proof. Just as a matter of of interest, we would like to point out that there are 3581 pairs of distinct triples $\{(r, s, t), (u, v, w)\}$ with $2 \leq r \leq s \leq t$ and $2 \leq u \leq v \leq w$ and $rst = uvw \leq 12,000,000$, satisfying the conclusions of Lemmas 8.3 and 8.5, with at most one of r, s, t being even. About half of these 3581 pairs can be eliminated using Corollary 8.8 (the ‘coprime’ test), and then most of the remaining pairs can be ruled out by Lemma 8.11 (using direct products). Just one such small pair cannot be ruled out in this way, namely $\{(17, 162, 459), (27, 34, 1377)\}$, but this can be eliminated by Corollary 8.10 (since $r = 17$ is coprime to $6, 27, 34/17$ and $1377/17$).

References

- [1] I. Agol, *The virtual Haken conjecture* (with an appendix by I. Agol, D. Groves and J. Manning), *Documenta Mathematica* **18** (2013), 1045–1087.
- [2] I. Agol, D. Groves and J. Manning, *Residual finiteness, QCERF and fillings of hyperbolic groups*, *Geometry and Topology* **13** (2009), 1043–1073.
- [3] I. Agol, D. D. Long and A.W. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, *Annals of Mathematics* **153** (2001), 599–621.
- [4] M. Aka, *Profinite completions and Kazhdan’s Property T*, *Groups, Geometry and Dynamics* **6** (2012), 221–229.
- [5] M. Aka, *Arithmetic groups with isomorphic finite quotients*, *Journal of Algebra* **352** (2012), 322–340.
- [6] J. Amorós, M. Burger, K. Corlette, D. Kotschick and D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, *Mathematical Surveys and Monographs* **44**, American Mathematical Society, Providence, RI, 1996.
- [7] G. Baumslag and J. E. Roseblade, *Subgroups of direct products of free groups*, *Journal London Mathematical Society* **30** (1984), 44–52.
- [8] G. Baumslag, A. Myasnikov and V. Remeslennikov, *Algebraic geometry over groups. I. Algebraic sets and ideal theory*, *J. Algebra* **219** (1999), 16–79.

- [9] W. Bosma, J. Cannon and C. Playoust, *The MAGMA Algebra System I: The User Language*, J. Symbolic Computation **24** (1997), 235–265.
- [10] E. Breuillard, T. Gelander, J. Souto and P. Storm, *Dense embeddings of surface groups*, Geometry and Topology **10** (2006), 1373–1389.
- [11] M. R. Bridson and F. Grunewald, *Grothendieck’s problems concerning profinite completions and representations of groups*, Annals of Mathematics **160** (2004), 359–373.
- [12] M. R. Bridson and A. W. Reid, *Nilpotent completions of groups, Grothendieck Pairs and four problems of Baumslag*, International Mathematics Research Notices IMRN 2015 no. 8, 2111–2140.
- [13] M.R. Bridson and H. Wilton, *Subgroup separability in residually free groups*, Mathematische Zeitschrift **260** (2008), 25–30.
- [14] D. Calegari, *Surface subgroups from homology*, Geometry and Topology **12** (2008), 1995–2007.
- [15] J. Cheeger and M. Gromov, *L^2 -cohomology and group cohomology*, Topology **25** (1986), 189–215.
- [16] T. Chinburg, D. D. Long and A. W. Reid, *Cusps of minimal non-compact arithmetic hyperbolic 3-orbifolds*, Pure and Applied Mathematics Quarterly **4** (2008), 1013–1031.
- [17] M. Conder, *Hurwitz groups: a brief survey*, Bulletin of the American Mathematical Society **23** (1990), 359–370.
- [18] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th ed., Springer, Berlin, 1980.
- [19] J. D. Dixon, E. W. Formanek, J. C. Poland and L. Ribes, *Profinite completions and isomorphic finite quotients*, Journal of Pure and Applied Algebra **23** (1982), 227–231.
- [20] B. Everitt, *Alternating quotients of Fuchsian groups*, Journal of Algebra **223** (2000), 457–476.
- [21] L. Funar, *Torus bundles not distinguished by TQFT invariants*, Geometry and Topology **17** (2013), 2289–2344.
- [22] C. McA. Gordon and H. Wilton, *On surface subgroups of doubles of free groups*, Journal London Mathematical Society **82** (2010), 17–31
- [23] F. Grunewald, A. Jaikin-Zapirain and P. A. Zalesskii, *Cohomological goodness and the profinite completion of Bianchi groups*, Duke Mathematical Journal **144** (2008), 53–72.
- [24] F. Grunewald and P. Zalesskii, *Genus for groups*, Journal of Algebra **326** (2011), 130–168.
- [25] W. Herfort and L. Ribes, *Torsion elements and centralizers in free products of profinite groups*, Journal für die reine und angewandte Mathematik **358** (1985), 155–161.
- [26] M. Jarden and A. Lubotzky, *Elementary equivalence of profinite groups*, Bulletin London Mathematical Society **40** (2008), 887–896.
- [27] J. Kahn and V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Annals of Mathematics **175** (2012), 1127–1190.
- [28] O. G. Kharlampovich and A. Myasnikov, *Elementary theory of free non-abelian groups*, Journal of Algebra **302** (2006), 451–552.
- [29] S. Kim and S. Oum, *Hyperbolic surface subgroups of one-ended doubles of free groups*, Journal of Topology **7** (2014), 927–947.

- [30] S. Kim and H. Wilton, *Polygonal words in free groups*, Quarterly Journal of Mathematics **63** (2012), 399–421.
- [31] M. Liebeck and A. Shalev, *Fuchsian groups, finite simple groups and representation varieties*, Inventiones mathematicae **159** (2005), 317–367.
- [32] D. D. Long, C. Maclachlan and A. W. Reid, *Splitting groups of signature $(1; n)$* , Journal of Algebra **185** (1996), 329–341.
- [33] D. D. Long and A. W. Reid, *Subgroup separability and virtual retractions of groups*, Topology **47** (2008), 137–159.
- [34] J. Lott, *Deficiencies of lattice subgroups of Lie groups*, Bulletin London Mathematical Society **31** (1999), 191–195.
- [35] W. Lück, *Approximating L^2 -invariants by their finite-dimensional analogues*, Geometric and Functional Analysis **4** (1994), 455–481.
- [36] W. Lück, *L^2 -invariants of regular coverings of compact manifolds and CW-complexes*, in *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, pp. 735–817.
- [37] A. M. Macbeath, *Generators of the fractional linear groups*, Proceedings of Symposia in Pure Mathematics **XII** (1969), 14–32.
- [38] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics **219**, Springer-Verlag, Berlin, 2003.
- [39] G. Margulis, *Discrete Subgroups of Semi-simple Lie Groups*, Ergebnisse der Mathematik und ihrer **17**, Springer-Verlag, Berlin, 1989.
- [40] D. B. McReynolds, *Peripheral separability and cusps of arithmetic hyperbolic orbifolds*, Algebraic and Geometric Topology **4** (2004), 721–755.
- [41] O. V. Melnikov and P. A. Zalesskii, *Subgroups of profinite groups acting on trees*, Mathematics of the USSR-Sbornik **63** (1989), 405–424.
- [42] T. Müller and J-C. Schlage-Puchta, *Character theory of symmetric groups, subgroup growth of Fuchsian groups, and random walks*, Advances in Mathematics **213** (2007), 919–982.
- [43] New York Group Theory Co-operative, *Problems*.
- [44] N. Nikolov and D. Segal, *On finitely generated profinite groups. I. Strong completeness and uniform bounds*, Annals of Mathematics **165** (2007), 171–238.
- [45] P. F. Pickel, *Finitely generated nilpotent groups with isomorphic quotients*, Transactions of the American Mathematical Society **160** (1971), 327–341.
- [46] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete **68**, Springer-Verlag, Berlin, 1972.
- [47] V. N. Remeslennikov, *Conjugacy of subgroups in nilpotent groups*, Algebra i Logika Sem. **6** (1967), 61–76. (Russian)
- [48] L. Ribes and P. A. Zalesskii, *Profinite Groups*, Ergebnisse der Mathematik und ihrer **40**, Springer-Verlag, Berlin, 2000.
- [49] G. P. Scott, *Subgroups of surface groups are almost geometric*, Journal London Mathematical Society **17** (1978), 555–565. See also *ibid Correction*: Journal London Mathematical Society **32** (1985), 217–220.
- [50] D. Segal, *Polycyclic Groups*, Cambridge University Press (1983).
- [51] Z. Sela, *Diophantine geometry over groups. VI. The elementary theory of a free group*, Geometric and Functional Analysis **16** (2006), 707–730.

- [52] J-P. Serre, *Galois Cohomology*, Springer-Verlag, Berlin, 1997.
- [53] P. F. Stebe, *A residual property of certain groups*, Proceedings of the American Mathematical Society **26** (1970), 37–42.
- [54] M. Suzuki, *Group Theory I*, Grundlehren der mathematischen Wissen **247**, Springer-Verlag, Berlin, 1982.
- [55] H. Wilton, *Hall's theorem for limit groups*, Geometric and Functional Analysis **18** (2008), 271–303.
- [56] D. T. Wise, *Cubulating small cancellation groups*, Geometric and Functional Analysis **14** (2004), 150–214.
- [57] D. T. Wise, *The structure of groups with a quasi-convex hierarchy*, 2012, preprint.