International Mathematics Research Notices Advance Access published January 7, 2013
D. D. Long and A. W. Reid (2013) "Constructing thin subgroups in SL(4, R)," International Mathematics Research Notices, rns281, 11 pages.
doi:10.1093/imrn/rns281

Constructing Thin Subgroups in $\mathrm{SL}(4, \mathbb{R})$

D. Darren Long[1] **and Alan W. Reid**[2]

[1]Department of Mathematics, University of California, Santa Barbara, CA 93106, USA and [2]Department of Mathematics, University of Texas, Austin, TX 78712, USA

Correspondence to be sent to: e-mail: long@math.ucsb.edu

We give a construction for new families of thin subgroups inside $\mathrm{SL}(4, \mathbf{R})$. In particular, we show that the fundamental group of a closed hyperbolic 3-manifold can be isomorphic to a thin subgroup of a lattice.

1 Introduction

Let G be a semi-simple Lie group and $\Gamma < G$ be a lattice. This paper is motivated by the attempt to understand the infinite index subgroup structure of Γ. In particular, to understand the possibilities for infinite index, finitely generated, freely indecomposable, Zariski dense subgroups of Γ.

The study of Zariski dense subgroups of semi-simple Lie groups has a long and rich history. Some highlights are the Borel Density Theorem, which establishes that a lattice in a semi-simple Lie group is Zariski dense, and works of Oh [17] and Venkataramana [23], which establish in certain cases that Zariski dense subgroups of a nonuniform lattice in a high rank Lie group are themselves lattices. In addition, there are many constructions (based on ping-pong) of free subgroups (or more generally subgroups that are free products), of semi-simple Lie groups, and lattices that are Zariski dense (see [16, 22] and references therein). Indeed, it is shown in [20] that, in a precise

Received June 11, 2012; Revised October 11, 2012; Accepted December 5, 2012

probabilistic sense, Zariski density is a generic property for subgroups of lattices in $\mathrm{SL}(n, \mathbf{Z})$ or $\mathrm{Sp}(2g, \mathbf{Z})$.

Following Sarnak (see [21]), a subgroup Δ of Γ is called *thin* if Δ has infinite index in Γ, and is Zariski dense. Since, as remarked upon above, it is by now quite standard to exhibit Zariski dense subgroups of lattices that are free products, the case of most interest is that the thin group Δ is finitely generated and does not decompose as a free product. In this paper, thin subgroups are understood to be freely indecomposable.

There has been much recent interest in the nature of thin subgroups of lattices, motivated in part by work on expanders, and in particular the so-called "affine sieve" of Bourgain et al. [5]. We cite [21] for a detailed survey; other recent works that study thin groups are [8–10, 14, 15, 20] to name but a few.

Thin subgroups appear quite difficult to exhibit since the Zariski dense condition makes any given subgroup hard to distinguish from a lattice. Some striking families of examples and nonexamples are offered in [15] (which generalizes an old example in [12]) and [14]. This paper continues our work in this direction and exhibits new examples of thin groups inside infinitely many nonuniform lattices in $\mathrm{SL}(4, \mathbf{R})$. In particular, we show that the fundamental group of a closed hyperbolic 3-manifold can be isomorphic to a thin subgroup. To our knowledge, these are the first such examples (see below for a precise statement).

The starting point of this paper is the construction of [7], where it is shown that for certain closed hyperbolic 3-manifolds, one can *flex* the faithful discrete representation into $\mathrm{SO}_0(3,1) < \mathrm{SL}(4, \mathbf{R})$; by which we mean that if one regards the hyperbolic structure as a strictly convex real projective structure, it can be deformed. In this way, we bring to bear deep results of Koszul [13] and Benoist [3, 4] to deduce that all of these deformations give rise to convex real projective structures. In particular, the holonomy representations are all discrete, faithful representations of the fundamental group in question. This part of the construction plays the role of the work of Choi–Goldman [6] used in [15]. One can then argue, using the results of [11], that all of these image groups, other than the discrete faithful $\mathrm{SO}_0(3,1)$-representation, are Zariski dense in $\mathrm{SL}(4, \mathbf{R})$ (see Theorem 2.1).

The argument to this point is quite general and applies to any flexible hyperbolic 3-manifold (even if these are apparently quite rare; see [7]), but some specialization is now necessary to ensure that the deformed group lies inside a lattice. To this end, we fix attention upon one particular closed hyperbolic 3-manifold, traditionally known as vol3. The lattices in question are described in Witte [24, Proposition 6.55]: One fixes a real quadratic number field L with ring of integers $\mathcal{O}_L$ and nontrivial Galois automorphism τ.

The lattices considered arise in the commensurability class of the Hermitean isometry groups of a diagonal form J, denoted by $\mathrm{SU}(J, \mathcal{O}_L, \tau)$. We refer to Theorem 2.3 for a precise description. After some slightly delicate technical work, we show the following theorem.

Theorem 1.1. For infinitely many real quadratic number fields L, the lattices $\mathrm{SU}(J, \mathcal{O}_L, \tau)$ contain a thin subgroup isomorphic to a subgroup of finite index in $\pi_1(\mathrm{vol3})$. □

We note that any such subgroup is not a free product (since it is the fundamental group of a closed hyperbolic 3-manifold) and must have infinite index in the lattice that contains it, since it is well known that $\pi_1(\mathrm{vol3})$ contains subgroups of finite index which map onto $\mathbf{Z}$. Indeed, one can show further that (see [18]) one can pass to a subgroup of finite index in $\pi_1(\mathrm{vol3})$ which is a surface bundle over the circle. This means one can find surface groups that are virtually normal subgroups in $\pi_1(\mathrm{vol3})$. Such a subgroup will have the same Zariski closure as $\pi_1(\mathrm{vol3})$, so that we have the following corollary.

Corollary 1.2. For infinitely many real quadratic number fields L, the lattices $\mathrm{SU}(J, \mathcal{O}_L, \tau)$ contain a thin surface subgroup. □

In this vein, the first author and M. Thistlethwaite (unpublished) have constructed a family of thin surface subgroups of $\mathrm{SL}(4, \mathbf{Z})$.

2 The Construction

As stated in Section 1, the group upon which our construction is based is the fundamental group of the closed hyperbolic 3-manifold known as vol3. The interest in this manifold is that one finds it is the simplest hyperbolic 3-manifold which is *flexible* in the sense of [7]; which is to say that the discrete faithful representation $\rho_\infty : \pi_1(\mathrm{vol3}) \to \mathrm{SO}_0(3, 1) < \mathrm{SL}(4, \mathbf{R})$ admits nontrivial deformations into $\mathrm{SL}(4, \mathbf{R})$. Of course, Mostow rigidity implies that these deformations cannot lie inside $\mathrm{SO}_0(3, 1)$. However, if one uses the Klein model for hyperbolic space, one can regard this hyperbolic structure as a strictly convex real projective and then remarkable results of Koszul [13] and Benoist [3, 4] show that all these deformations correspond to holonomy deformations *qua* strictly convex real projective structures and, in particular, they are all discrete and faithful representations of the group $\pi_1(\mathrm{vol3})$ into $\mathrm{SL}(4, \mathbf{R})$. Moreover, we also have the following theorem.

Theorem 2.1. Let ρ be irreducible and a nontrivial (i.e., nonconjugate) deformation of ρ_∞.

Then the image $\rho(\pi_1(\text{vol3}))$ is Zariski dense in $\mathrm{SL}(4, \mathbf{R})$. □

Proof. This is based upon a theorem due to Benoist. Theorem 1.1 of [3] describes the various possibilities for convex real projective structures in our setting. The representation ρ is irreducible, so that the convex set C associated to the real projective ρ-structure cannot split invariantly as a product of convex sets $C_1 \times C_2$.

The statement of Benoist's Theorem 1.1 now becomes either $\rho(\pi_1(\text{vol3}))$ has Zariski closure $\mathrm{SL}(4, \mathbf{R})$, or that C is *homogeneous*, which is to say that $\mathrm{Aut}(C)$ acts transitively on C. However, we may rule out this case as follows.

The action of $\mathrm{Aut}(C)$ is by isometries of the Hilbert metric on C. Now it is a standard fact (e.g., one can base an argument on [1, Theorem 2.17]) that one can associate to any Finsler metric a Riemannian metric which is sufficiently canonical that the action of $\mathrm{Aut}(C)$ is by isometries of this Riemannian metric; that is to say, we have assigned to C a Riemannian metric making it into a homogeneous space. This makes $C/\rho(\pi_1(\text{vol3}))$ into a closed manifold with a homogeneous metric. Such homogeneous metrics have been classified by Thurston, and the only possibility is that the homogeneous metric on $C/\rho(\pi_1(\text{vol3}))$ is a multiple of the hyperbolic metric.

However, we now claim that this contradicts the fact that the representation has been flexed away from the canonical representation. We argue as follows: We have identified $\mathrm{Aut}(C)$ as a subgroup of $\mathrm{SO}_0(3,1)$ and since the only transitive nonsoluble Lie subgroup of $\mathrm{SO}_0(3,1)$ is the whole group, this shows that $\mathrm{Aut}(C) \cong \mathrm{SO}_0(3,1)$.

However, all representations of $\mathrm{SO}_0(3,1)$ into $\mathrm{SL}(4, \mathbf{R})$ preserve a nondegenerate invariant bilinear form (see, e.g., [11, p. 205 and Example 3, p. 198]). This form cannot be definite since $\mathrm{Aut}(C)$ contains an infinite discrete subgroup, and cannot have signature (2, 2) since $\mathrm{SO}_0(3,1)$ and $\mathrm{SO}(2,2)$ are not locally isomorphic. It follows that we must have signature (3, 1) and therefore up to conjugacy in $\mathrm{SL}(4, \mathbf{R})$ the representation is equivalent to the standard one. We deduce that the flexed representation ρ can be conjugated into $\mathrm{SO}_0(3,1)$.

Finally, we conclude that this contradicts Mostow rigidity. The reason is as follows: The manifold $C/\rho(\pi_1(\text{vol3}))$ is a $K(\pi, 1)$ and therefore $H_3(C/\rho(\pi_1(\text{vol3}))) = \mathbf{Z}$. This implies that $C/\rho(\pi_1(\text{vol3}))$ is a closed 3-manifold and we deduce that $\rho(\pi_1(\text{vol3}))$ would be a lattice in $\mathrm{Aut}(C) \cong \mathrm{SO}_0(3,1)$. Mostow rigidity implies that ρ is conjugate to the standard representation, contradicting that ρ is a nontrivial flexing of ρ_∞.

Hence, C cannot be homogeneous and therefore $\rho(\pi_1(\mathrm{vol3}))$ is Zariski dense in $\mathrm{SL}(4, \mathbf{R})$, as required. ∎

We now collect some details about vol3. It is an arithmetic hyperbolic 3-manifold with volume the same as that of a regular ideal simplex (i.e., approximately 1.01494160640965...), and $H_1(\mathrm{vol3}, \mathbf{Z}) = \mathbf{Z}_6 \oplus \mathbf{Z}_3$. It is known that vol3 is non-Haken and indeed arithmetic methods show that $\rho_\infty(\pi_1(\mathrm{vol3}))$ contains no nonelementary Fuchsian subgroups (see [18] for details).

The fundamental group has presentation

$$\langle a, b \mid aabbABAbb,\ \ aBaBabaaab\rangle,$$

where $A = a^{-1}$ and $B = b^{-1}$.

It will be technically slightly easier to work with an orbifold $Q = \mathrm{vol3}/\langle u\rangle$ which is four-fold covered by vol3. We denote by Γ the orbifold fundamental group of Q. Note that a representation of Γ is discrete and faithful only if it is discrete and faithful when restricted to vol3, so that it suffices to work with Γ.

One finds that Γ is generated by two elements of finite order u and c (see [7]). The group $\pi_1(\mathrm{vol3})$ is recovered as $a = u^2 \cdot c$ and $b = (a \cdot u \cdot a)^{-1} \cdot u$.

We shall need quite a detailed understanding of the representations of Γ; initially, we use the version presented in [7], where, after some mild conjugacy, one finds that the 4-dimensional representations are given by

$$\rho_v(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\dfrac{v^2-4}{v^2+8}} & 1 \\ 0 & 0 & -\dfrac{2(v^2+2)}{v^2+8} & -\sqrt{\dfrac{v^2-4}{v^2+8}} \end{pmatrix}$$

and

$$\rho_v(c) = \begin{pmatrix} \left(v+\sqrt{v^2+8}\right)/4 & 0 & \left(4-v^2-v\sqrt{v^2+8}\right)/8 & 0 \\ 0 & \left(v-\sqrt{v^2+8}\right)/4 & 0 & \left(-4+v^2-v\sqrt{v^2+8}\right)/8 \\ 1 & 0 & \left(-v-\sqrt{v^2+8}\right)/4 & 0 \\ 0 & -1 & 0 & \left(-v+\sqrt{v^2+8}\right)/4 \end{pmatrix}.$$

The discrete faithful representation occurs at $v=2$ and the deformations as a convex real projective manifold are parameterized by real $v\in[2,\infty)$; the observations above show that all these representations are discrete and faithful representations of Γ.

One crucial property we shall need is the following proposition.

Proposition 2.2. For every $v\in\mathbf{Z}$ with $v\geq 2$, the traces of ρ_v lie in the ring of integers $\mathcal{O}_{\mathbf{Q}(\sqrt{v^2-4})}$. □

Proof. This is a computation; we sketch the proof. One constructs a list of 16 group elements $\{M_j\}$ in Γ that are linearly independent in $M(4,\mathbf{R})$: $\{\mathrm{id}, uc, u^2c, c, cuc, cu^2c, (uc)^2, u^2cuc, ucu^2c, u^2cu^2c, cucuc, cu^2cuc, cucu^2c, cu^2cu^2c, u^2cucuc, ucucu^2cuc\}$.

This is then used to construct a 16-dimensional representation of Γ by using left multiplication on this basis. (A file with this representation has been placed at [25].) One finds that, for the given values of v, the entries for this representation are elements of $\mathcal{O}_{\mathbf{Q}(\sqrt{v^2-4})}$.

The proof of Proposition 2.2 is completed as follows. One can check by hand that the traces of the elements M_j are all integers. Now the image of the general element $\gamma\in\Gamma$ for the left regular representation constructed above has all its entries in $\mathcal{O}_{\mathbf{Q}(\sqrt{v^2-4})}$, and since the first element of the list is the identity, the first column of this matrix is 16 integers $\{c_j\}$, which express the fact that

$$g=\sum_j c_jM_j.$$

Since the trace is linear, $\mathrm{tr}(g)=\sum_j c_j\,\mathrm{tr}(M_j)$ is an integer, as required. ■

The next phase of the proof is to show that, for infinitely values of $v\in\mathbf{Z}$, the images $\rho_v(\Gamma)$ lie in a lattice inside $\mathrm{SL}(4,\mathbf{R})$. The lattices in question are constructed from [24, Theorem 6.55] by a rather general construction which involves L, a real quadratic extension of $\mathbf{Q}$ and D, a central simple division algebra of degree d over L. However, in our situation we may assume that $D=L$, and we state only this special case.

Before stating the theorem, recall that if L is a real quadratic extension of $\mathbf{Q}$, and $A\in\mathrm{SL}(4,L)$, we denote by A^* the matrix obtained by taking the transpose of the matrix obtained from A by applying τ (the nontrivial Galois automorphism) to all its entries.

Theorem 2.3. Suppose that L is a real quadratic extension of $\mathbf{Q}$, with Galois automorphism τ. Suppose that $b_1, \ldots, b_4$ are nonzero elements of $\mathbf{Z}$. Setting $J = \mathrm{diag}(b_1, \ldots, b_4)$, then the group

$$\mathrm{SU}(J, \mathcal{O}_L, \tau) = \{A \in \mathrm{SL}(4, \mathcal{O}_L) \mid A^* J A = J\}$$

is a lattice in SL(4, **R**). □

We also note from [24, Proposition 6.55] that, in the case being considered here (when $D = L$), the corresponding forms will represent zero nontrivially, and so the lattices produced are nonuniform.

It is not immediate how to apply Theorem 2.3 directly, since as it stands the entries of ρ_v lie in a biquadratic field. However, one can show that the representation ρ_v is conjugate to the representation below with $v = (r - 1/r)/\sqrt{2}$.

$$\phi_r(u) = \begin{pmatrix} 0 & -\dfrac{(1+r^2)(2+2r^2-\sqrt{2(1-10r^2+r^4)})}{1+14r^2+r^4} & 0 & 0 \\ \dfrac{2+2r^2+\sqrt{2(1-10r^2+r^4)}}{2+2r^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\phi_r(c) = \begin{pmatrix} \dfrac{1-r^2}{4\sqrt{2}r} & -\dfrac{1+r^2}{4r} & -1 & 0 \\ -\dfrac{1+14r^2+r^4}{8(r+r^3)} & \dfrac{1-r^2}{4\sqrt{2}r} & 0 & 1 \\ \dfrac{1}{16}(-10+1/r^2+r^2) & \dfrac{-1+r^4}{8\sqrt{2}r^2} & \dfrac{-1+r^2}{4\sqrt{2}r} & -\dfrac{1+r^2}{4r} \\ -\dfrac{(-1+r^2)(1+14r^2+r^4)}{16\sqrt{2}r^2(1+r^2)} & \dfrac{1}{16}(10-1/r^2-r^2) & -\dfrac{1+14r^2+r^4}{8(r+r^3)} & \dfrac{-1+r^2}{4\sqrt{2}r} \end{pmatrix}.$$

The utility of this version of the representation is that if one chooses r to be a unit $\alpha \in \mathbf{Z}[\sqrt{2}]$ with the property that the Galois automorphism carries $\alpha \to 1/\alpha$, then it is a simple exercise to see that the entries of $\phi_\alpha(\Gamma)$ lie in a real quadratic field $\mathbf{Q}(\sqrt{2\alpha^2 - 20 + 2/\alpha^2})$; indeed $\phi_\alpha(c)$ is a rational matrix. In this notation, v is the integer $(\alpha - 1/\alpha)/\sqrt{2}$; the simplest case $\alpha = 3 + 2\sqrt{2}$ gives $v = 4$.

The passage to a commensurable integral representation is achieved by the following theorem.

Theorem 2.4. Suppose that $\Gamma < \mathrm{SL}(4, k)$ is a finitely generated group with the property that $\mathrm{tr}(\gamma) \in \mathcal{O}_k$ for every $\gamma \in \Gamma$.

Then Γ has a subgroup of finite index contained in $\mathrm{SL}(4, \mathcal{O}_k)$. □

Proof. Consider

$$\mathcal{O}\Gamma = \{\Sigma a_i \gamma_i \mid a_i \in \mathcal{O}_k, \gamma_i \in \Gamma\},$$

where the sums are finite. It is shown in [2] (see Proposition 2.2 and Corollary 2.3), that $\mathcal{O}\Gamma$ is an order of a central simple subalgebra $B \subset M(4, k)$ defined over k. Now, while $\mathcal{O}\Gamma$ need not be an order in $M(4, k)$, it is known that it is contained in some maximal order $\mathcal{D}$ of $M(4, k)$ (cf. [15, Exercise 5 and Proof of Lemma 2.3; 19, p. 131).

Now it is a standard fact that the groups of elements of norm 1 in orders contained in $M(4, k)$ are commensurable (since the intersection of two orders is an order and the unit groups of orders will be irreducible lattices in $\mathrm{SL}(4, \mathbf{R}) \times \mathrm{SL}(4, \mathbf{R})$; see [24, Chapter 15I]). In particular, $\mathrm{SL}(4, \mathcal{O}_k)$ and $\mathcal{D}^1$ are commensurable. Let $\Delta = \mathrm{SL}(4, \mathcal{O}_k) \cap \mathcal{D}^1$, which has finite index in both groups. Then $\Gamma \leq \mathcal{D}^1$, so that $\Gamma \cap \Delta$ has finite index in Γ and lies inside $\mathrm{SL}(4, \mathcal{O}_k)$, as required. ■

For the specializations α above, one can perform a computation to find a nonsingular Hermitean matrix J which exhibits $\phi_\alpha(\Gamma)$ as a subgroup of $\mathrm{SU}(J, L, \tau)$, where L is the real quadratic field $\mathbf{Q}\left(\sqrt{2\alpha^2 - 20 + 2/\alpha^2}\right)$. Hence, by Theorem 2.4, we may pass to a subgroup H of finite index in Γ for which $\phi_\alpha(H)$ lies inside $\mathrm{SU}(J, \mathcal{O}_L, \tau)$. By using the Gram–Schmidt process, one can diagonalize J over L, and by scaling assume that the diagonal entries lie in $\mathcal{O}_L$, hence in $\mathbf{Z}$. The main theorem will now follow from the following simple lemma.

Lemma 2.5. Suppose that the forms J_1 and J_2 are $\mathrm{GL}(4, L)$-equivalent.

Then, after a conjugacy in $\mathrm{GL}(4, L)$, the groups $\Gamma_1 = \mathrm{SU}(J_1, \mathcal{O}_L, \tau)$ and $\Gamma_2 = \mathrm{SU}(J_2, \mathcal{O}_L, \tau)$ are commensurable. □

Proof. If $J_1 = M^* J_2 M$ say, and $A \in \Gamma_2$, then $M^{-1} \cdot A \cdot M$ is an isometry of J_1, but need not have entries in $\mathcal{O}_L$ (since $\det(M)$ need not be a unit). However, this may be rectified by passing to a sufficiently deep congruence subgroup $K \leq \Gamma_2$. Then $M^{-1}KM \leq \Gamma_1$ necessarily of finite index by volume considerations. ∎

The result of Theorem 1.1 now follows. Taking $r = \alpha$, a unit of the required form, the representation $\phi_\alpha(\Gamma)$ is a discrete, faithful representation of Γ into SL(4, **R**) which has image whose entries lie in $L = \mathbf{Q}\left(\sqrt{v^2-4}\right)$, where $v = (\alpha - 1/\alpha)/\sqrt{2}$. Theorem 2.1 shows that this subgroup is Zariski dense in SL(4, **R**).

By Theorem 2.4, one can pass to a subgroup of finite index which has entries in $\mathcal{O}_L$. A computation finds a τ Hermitean form J for which this subgroup lies inside $\mathrm{SU}(J, \mathcal{O}_L, \tau)$, where τ is the nontrivial Galois automorphism of L. The Gram–Schmidt process converts this form to a diagonal **Z**-form J', so that an application of Lemma 2.5 shows that by passing to a further subgroup of finite index, $\phi_\alpha(\Gamma)$ is commensurable with a subgroup of $\mathrm{SU}(J', \mathcal{O}_L, \tau)$; this is a nonuniform lattice by Theorem 2.3 and the discussion following it.

Finally, we note that the image group $\phi_\alpha(\Gamma)$ cannot be commensurable with a subgroup of finite index in $\mathrm{SU}(J', \mathcal{O}_L, \tau)$, since as pointed out in Section 1, $\pi_1(\mathrm{vol3})$ contains subgroups of finite index that map onto **Z**. Hence, this subgroup is thin, as claimed.

3 An Example

The simplest example is when $v = 4$. It is easily checked that the representation in Section 2 is conjugate to

$$\rho_4(u) = \begin{pmatrix} 0 & -1 & -4+3\sqrt{3} & -1+2\sqrt{3} \\ 1 & 0 & -2+\sqrt{3} & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_4(c) = \begin{pmatrix} 0 & 0 & -1+\sqrt{3} & 0 \\ 0 & 0 & 0 & -1+\sqrt{3} \\ (1+\sqrt{3})/2 & 0 & 0 & 0 \\ 0 & (1+\sqrt{3})/2 & 0 & 0 \end{pmatrix}.$$

This is not integral, but there is a surjection from $\pi_1(\mathrm{vol3})$ to the dihedral group with 10 elements where one sends a to a reflection and b to a rotation. A direct calculation shows that the kernel of this map consists of elements whose entries lie in $\mathbf{Z}[\sqrt{3}]$. One can also check that a nondegenerate τ-Hermitean form for the image of ρ_4 is

$$J = \begin{pmatrix} 2 & 0 & 2-2\sqrt{3} & -2\sqrt{3} \\ 0 & 2 & 6-4\sqrt{3} & 2-2\sqrt{3} \\ 2+2\sqrt{3} & 6+4\sqrt{3} & -4 & 0 \\ 2\sqrt{3} & 2+2\sqrt{3} & 0 & -4 \end{pmatrix},$$

which in turn can be checked as being equivalent to the diagonal form $\mathrm{diag}(1, 1, 1, -5)$.

Funding

This work was partially supported by the National Science Foundation (to D.D.L. and A.W.R.).

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