PROFINITE PROPERTIES OF DISCRETE GROUPS

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1 Introduction

This paper is based on a series of 4 lectures delivered at Groups St Andrews 2013. The main theme of the lectures was *distinguishing finitely generated residually finite groups by their finite quotients*. The purpose of this paper is to expand and develop the lectures.

The paper is organized as follows. In §2 we collect some questions that motivated the lectures and this article, and in §3 discuss some examples related to these questions. In §4 we recall profinite groups, profinite completions and the formulation of the questions in the language of the profinite completion. In §5, we recall a particular case of the question of when groups have the same profinite completion, namely Grothendieck's question. In §6 we discuss how the methods of L^2-cohomology can be brought to bear on the questions in §2, and in §7, we give a similar discussion using the methods of the cohomology of profinite groups. In §8 we discuss the questions in §2 in the context of groups arising naturally in low-dimensional topology and geometry, and in §9 discuss parafree groups. Finally in §10 we collect a list of open problems that may be of interest.

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2 The motivating questions

We begin by recalling some terminology. A group Γ is said to be *residually finite* (resp., *residually nilpotent*, *residually-p*, *residually torsion-free-nilpotent*) if for each non-trivial $\gamma \in \Gamma$ there exists a finite group (resp., nilpotent group, p-group, torsion-free-nilpotent group) Q and a homomorphism $\phi : \Gamma \to Q$ with $\phi(\gamma) \neq 1$.

2.1. If a finitely-generated group Γ is residually finite, then one can recover any finite portion of its Cayley graph by examining the finite quotients of the group. It is therefore natural to wonder whether, under reasonable hypotheses, the set

$$\mathcal{C}(\Gamma) = \{G : G \text{ is a finite quotient of } \Gamma\}$$

might determine Γ up to isomorphism.

Assuming that the groups considered are residually finite is a natural condition to impose, since, first, this guarantees a rich supply of finite quotients, and secondly, one can always form the free product $\Gamma * S$ where S is a finitely generated infinite simple group, and then, clearly $\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma * S)$. Henceforth, unless otherwise stated, all groups considered will be residually finite.

The basic motivating question of this work is the following due to Remesselenikov:

Question 1: *If F_n is the free group of rank n, and Γ is a finitely-generated, residually finite group, then does $\mathcal{C}(\Gamma) = \mathcal{C}(F_n)$ imply that $\Gamma \cong F_n$?*

This remains open at present, although in this paper we describe progress on this question, as well as providing structural results about such a group Γ (should it exist) as in Question 1.

Following [31], we define *the genus* of a finitely generated residually finite group Γ to be:

$$\mathcal{G}(\Gamma) = \{\Delta : \mathcal{C}(\Delta) = \mathcal{C}(\Gamma)\}.$$

This definition is taken, by analogy with the theory of quadratic forms over $\mathbf{Z}$ where two integral quadratic forms can be locally equivalent (i.e., at all places of $\mathbf{Q}$), but not globally equivalent over $\mathbf{Z}$.

Question 2: *Which finitely generated (respectively, finitely presented) groups Γ have $\mathcal{G}(\Gamma) = \{\Gamma\}$?*

Question 3: *Which finitely generated (respectively, finitely presented) groups Γ have $|\mathcal{G}(\Gamma)| > 1$?*

Question 4: *How large can $|\mathcal{G}(\Gamma)|$ be for finitely generated (resp., finitely presented) groups?*

Question 5: *What group theoretic properties are shared by (resp., are different for) groups in the same genus?*

In addition, if $\mathcal{P}$ is a class of groups, then we define

$$\mathcal{G}(\Gamma, \mathcal{P}) = \{\Delta \in \mathcal{P} : \mathcal{C}(\Delta) = \mathcal{C}(\Gamma)\},$$

and can ask the same questions upon restricting to groups in $\mathcal{P}$.

2.2. Rather than restricting the class of groups in a genus, we can ask to distinguish finitely generated groups by restricting the quotient groups considered. A particularly interesting case of this is the following. Note first that, a group Γ is residually nilpotent if and only if $\bigcap \Gamma_n = 1$, where Γ_n, the n-th term of the *lower central series* of Γ, defined inductively by setting $\Gamma_1 = \Gamma$ and defining $\Gamma_{n+1} = \langle\, [x, y] : x \in \Gamma_n, y \in \Gamma \,\rangle$.

Two residually nilpotent groups Γ and Λ are said to have the same *nilpotent genus* if they have the same lower central series quotients; i.e., $\Gamma/\Gamma_c \cong \Lambda/\Lambda_c$ for all $c \geq 1$. Residually nilpotent groups with the same nilpotent genus as a free group are termed *parafree*. In [10] Gilbert Baumslag surveyed the state of the art concerning groups of

the same nilpotent genus with particular emphasis on the nature of parafree groups. We will discuss this in more detail in §9 below.

3 Some examples

We begin with a series of examples where one can say something about Questions 1–4.

3.1. We first prove the following elementary result.

Proposition 3.1 *Let Γ be a finitely generated abelian group, then $\mathcal{G}(\Gamma) = \{\Gamma\}$.*

Proof Suppose first that $\Delta \in \mathcal{G}(\Gamma)$ and Δ is non-abelian. We may therefore find a commutator $c = [a, b]$ that is non-trivial. Since Δ is residually finite there is a homomorphism $\phi : \Delta \to Q$, with Q finite and $\phi(c) \neq 1$. However, $\Delta \in \mathcal{G}(\Gamma)$ and so Q is abelian. Hence $\phi(c) = 1$, a contradiction.

Thus Δ is abelian. We can assume that $\Gamma \cong \mathbf{Z}^r \oplus T_1$ and $\Delta \cong \mathbf{Z}^s \oplus T_2$, where T_i $(i = 1, 2)$ are finite abelian groups. It is easy to see that $r = s$, for if $r > s$ say, we can choose a large prime p such that p does not divide $|T_1||T_2|$, and construct a finite quotient $(\mathbf{Z}/p\mathbf{Z})^r$ that cannot be a quotient of Δ.

In addition if T_1 is not isomorphic to T_2, then some invariant factor appears in T_1 say, but not in T_2. One can then construct a finite abelian group that is a quotient of T_1 (and hence Γ_1) but not of Γ_2. □

Note that the proof of Proposition 3.1 also proves the following.

Proposition 3.2 *Let Γ be a finitely generated group, and suppose that $\Delta \in \mathcal{G}(\Gamma)$. Then $\Gamma^{\mathrm{ab}} \cong \Delta^{\mathrm{ab}}$. In particular $b_1(\Gamma) = b_1(\Delta)$.*

3.2. Remarkably, moving only slightly beyond $\mathbf{Z}$ to groups that are virtually $\mathbf{Z}$, the situation is dramatically different. The following result is due to Baumslag [9]. We include a sketch of the proof.

Theorem 3.3 *There exists non-isomorphic meta-cyclic groups Γ_1 and Γ_2 for which $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$. Indeed, both of these groups are virtually $\mathbf{Z}$ and defined as extensions of a fixed finite cyclic group F by $\mathbf{Z}$.*

Sketch Proof What Baumslag actually proves in [9] is the following, and this is what we sketch a proof of:

> $(*)$ *Let F be a finite cyclic group with an automorphism of order n, where n is different from 1, 2, 3, 4 and 6. Then there are at least two non-isomorphic cyclic extensions of F, say Γ_1 and Γ_2 with $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

Recall that the automorphism group of a finite cyclic group of order m is an abelian group of order $\phi(m)$. So in $(*)$ we could take F to be a cyclic group of order 11, which has an automorphism of order 5.

Now let $F = \langle a \rangle$ be a cyclic group of order m, and assume that it admits an automorphism α of order n as in $(*)$. Assume that $\alpha(a) = a^r$. Now some elementary number theory (using that $\phi(m) > 2$ by assumption) shows that we can find an integer ℓ such that $(\ell, n) = 1$, and

$$\text{(i) } \alpha^\ell \neq \alpha, \text{ and (ii) } \alpha^\ell \neq \alpha^{-1}.$$

Now define $\Gamma_1 = \langle\, a, b \mid a^m = 1, b^{-1}ab = a^r \,\rangle$ to be the split extension of F induced by α and $\Gamma_2 = \langle\, a, c \mid a^m = 1, c^{-1}ac = a^{r^\ell} \,\rangle$ be the split extension of F induced by α^ℓ. The key claims to be established are that Γ_1 and Γ_2 are non-isomorphic, and that they have the same genus.

That the groups are non-isomorphic can be checked directly as follows. If $\theta : \Gamma_1 \to \Gamma_2$ is an isomorphism, then θ must map the set of elements of finite order in Γ_1 to those in Γ_2; that is to say θ preserves F, and so induces an automorphism of F. Thus $\theta(a) = a^s$ where $(s, m) = 1$. Moreover since the quotients $\Gamma_i/F \cong \mathbf{Z}$ for $i = 1, 2$, it follows that $\theta(b) = c^\epsilon a^t$ where $\epsilon = \pm 1$ and t is an integer. Now consider $\theta(a^r)$. When $\theta(b) = ca^t$ we get:

$$\alpha(a^s) = a^{rs} = \theta(a^r) = \theta(bab^{-1}) = (ca^t)a^s(ca^t)^{-1} = \alpha^\ell(a^s),$$

and it follows that $\alpha = \alpha^\ell$. A similar argument holds when $\theta(b) = c^{-1}a^t$ to show $\alpha^{-1} = \alpha^\ell$, both of which are contradictions to (ii) above.

We now discuss proving that the groups are in the same genus. Setting $P = \Gamma_1 \times \mathbf{Z}$, Baumslag [9] shows that P is isomorphic to $\Gamma_2 \times \mathbf{Z}$. That Γ_1 and Γ_2 have the same genus now follows from a result of Hirshon [34] (see also [9]) where it is shown that (see Theorem 9 of [34]):

Proposition 3.4 *Suppose that A and B are groups with $A \times \mathbf{Z} \cong B \times \mathbf{Z}$, then $\mathcal{C}(A) = \mathcal{C}(B)$.*

3.3. The case of nilpotent groups more generally is well understood due to work of Pickel [52]. We will not discuss this in any detail, other than to say that, in [52] it is shown that for a finitely generated nilpotent group Γ, $\mathcal{G}(\Gamma)$ consists of a finite number of isomorphism classes of nilpotent groups, and moreover, examples where the genus can be made arbitrarily large are known (see for example [58] Chapter 11). Similar results are also known for polycyclic groups (see [29] and [58]).

3.4. From the perspective of this article, more interesting examples where the genus has cardinality greater than 1 (although still finite) are given by examples of lattices in semi-simple Lie groups. We refer the reader to [4] and [5] for details but we will provide a sketch of some salient points.

Let Γ be a lattice in a semi-simple Lie group, for example, in what follows we shall take $\Gamma = \mathrm{SL}(n, R_k)$ where R_k denotes the ring of integers in a number field k. A natural, obvious class of finite quotients of Γ, are those of the form $\mathrm{SL}(n, R_k/I)$ where $I \subset R_k$ is an ideal. Let π_I denote the reduction homomorphism $\Gamma \to \mathrm{SL}(n, R_k/I)$, and $\Gamma(I)$ the kernel. Note that by Strong Approximation for SL_n (see [53] Chapter 7.4 for example) π_I is surjective for all I. A *congruence subgroup* of Γ is any subgroup

$\Delta < \Gamma$ such that $\Gamma(I) < \Delta$ for some I. A group Γ is said to have the *Congruence Subgroup Property* (henceforth abbreviated to CSP) if every subgroup of finite index is a congruence subgroup.

Thus, if Γ has CSP, then $\mathcal{C}(\Gamma)$ is known precisely, and in effect, to determine $\mathcal{C}(\Gamma)$ is reduced to number theory. Expanding on this, since R_k is a Dedekind domain, any ideal I factorizes into powers of prime ideals. If $I = \prod \mathcal{P}_i^{a_i}$, then it is known that $\mathrm{SL}(n, R_k/I) = \prod \mathrm{SL}(n, R_k/\mathcal{P}_i^{a_i})$. Thus the finite groups that arise as quotients of $\mathrm{SL}(n, R_k)$ are determined by those of the form $\mathrm{SL}(n, R_k/\mathcal{P}_i^{a_i})$. Hence we are reduced to understanding how a rational prime p behaves in the extension $k/\mathbf{Q}$. This idea, coupled with the work of Serre [59] which has shed considerable light on when Γ has CSP, allows construction of non-isomorphic lattices in the same genus.

Example: Let $k_1 = \mathbf{Q}(\sqrt[8]{3^7})$ and $k_2 = \mathbf{Q}(\sqrt[8]{48})$. Let $\Gamma_1 = \mathrm{SL}(n, R_{k_1})$ and $\Gamma_2 = \mathrm{SL}(n, R_{k_2})$ ($n \geq 3$). Then Γ_1 and Γ_2 have CSP (by [59]), are non-isomorphic (by rigidity) and $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$. The reason for the last statement is that the fields k_1 and k_2 are known to be *adelically equivalent* (see [36]); i.e. their Adele rings are isomorphic. This can be reformulated as saying that if V_i ($i = 1, 2$) are the sets of valuations associated to the prime ideals in k_1 and k_2, then there is a bijection $\phi : V_1 \to V_2$ such that for all $\nu \in V_1$ we have isomorphisms $(k_1)_\nu \cong (k_2)_{\phi(\nu)}$. This has, as a consequence, the desired identical splitting behavior of rational primes in k_1 and k_2.

3.5. Unlike in the previous subsections, there are recent examples of Bridson [14] of finitely presented groups Γ for which $\mathcal{G}(\Gamma)$ is infinite. This will be discussed further in §5.1.

4 Profinite methods

An important reformulation of the discussion in §2 uses the language of profinite groups. In particular, the language of profinite completions is a particularly convenient formalism for organizing finite quotients of a discrete group. For completeness we provide some discussion of profinite groups and profinite completions of discrete groups. We refer the reader to [56] for a more detailed account of the topics covered here.

4.1. A *directed set* is a partially ordered set I such that for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$. An *inverse system* is a family of sets $\{X_i\}_{\{i \in I\}}$, where I is a directed set, and a family of maps $\phi_{ij} : X_i \to X_j$ whenever $i \geq j$, such that:

- $\phi_{ii} = id_{X_i}$;
- $\phi_{ij}\phi_{jk} = \phi_{ik}$, whenever $i \geq j \geq k$.

Denoting this system by (X_i, ϕ_{ij}, I), the *inverse limit* of the inverse system (X_i, ϕ_{ij}, I) is the set

$$\varprojlim X_i = \left\{ (x_i) \in \prod_{i \in I} X_i \;\middle|\; \phi_{ij}(x_i) = x_j, \text{ whenever } i \geq j \right\}.$$

We record the following standard facts about the inverse limit (see [56] Chapter 1 for further details):

(i) Let (X_i, ϕ_{ij}, I) be an inverse system of non-empty compact, Hausdorff, totally disconnected topological spaces (resp. topological groups) over the directed set I, then $\varprojlim X_i$ is a non-empty, compact, Hausdorff, totally disconnected topological space (resp. topological group).

(ii) Let (X_i, ϕ_{ij}, I) be an inverse system. A subset $J \subset I$ is defined to be *cofinal*, if for each $i \in I$, there exists $j \in J$ with $j \geq i$. If J is cofinal we may form an inverse system (X_j, ϕ_{ij}, J) obtained by omitting those $i \in I \setminus J$. The inverse limit $\varprojlim X_j$ can be identified with the image of $\varprojlim X_i$ under the projection map $\prod_{i\in I} X_i$ onto $\prod_{j\in J} X_j$.

4.2. Returning to the world of group theory, let Γ be a finitely generated group (not necessarily residually finite for this discussion), and let $\mathcal{N}$ denote the collection of all finite index normal subgroups of Γ. Note that $\mathcal{N}$ is non-empty as $\Gamma \in \mathcal{N}$, and we can make $\mathcal{N}$ into directed set by declaring that

$$\text{For } M, N \in \mathcal{N},\ M \leq N \text{ whenever } M \textit{ contains } N.$$

In this case, there are natural epimorphisms $\phi_{NM} : \Gamma/N \to \Gamma/M$, and the inverse limit of the inverse system $(\Gamma/N, \phi_{NM}, \mathcal{N})$ is denoted $\widehat{\Gamma}$ and defined to be to the *profinite completion* of Γ.

Note that there is a natural map $\iota : \Gamma \to \widehat{\Gamma}$ defined by

$$g \mapsto (gN) \in \varprojlim \Gamma/N,$$

and it is easy to see that ι is injective if and only if Γ is residually finite.

An alternative, perhaps more concrete way of viewing the profinite completion is as follows. If, for each $N \in \mathcal{N}$, we equip each Γ/N with the discrete topology, then $\prod\{\Gamma/N : N \in \mathcal{N}\}$ is a compact space and $\widehat{\Gamma}$ can be identified with $\overline{j(\Gamma)}$ where $j : \Gamma \to \prod\{\Gamma/N : N \in \mathcal{N}\}$ is the map $g \mapsto (gN)$.

4.3. From §4.1, $\widehat{\Gamma}$ is a compact topological group, and so a subgroup U is open if and only if it is closed of finite index. In addition, a subgroup $H < \widehat{\Gamma}$ is closed if and only if it is the intersection of all open subgroups of $\widehat{\Gamma}$ containing it. More recently, it is a consequence of a deep theorem of Nikolov and Segal [50] that if Γ is a finitely generated group, then every finite index subgroup of $\widehat{\Gamma}$ is open. Thus a consequence of this is the following elementary lemma (in which $\mathrm{Hom}(G, Q)$ denotes the set of homomorphisms from the group G to the group Q, and $\mathrm{Epi}(G, Q)$ denotes the set of epimorphisms).

Lemma 4.1 *Let Γ be a finitely-generated group and let $\iota : \Gamma \to \widehat{\Gamma}$ be the natural map to its profinite completion. Then, for every finite group Q, the map $\mathrm{Hom}(\widehat{\Gamma}, Q) \to \mathrm{Hom}(\Gamma, Q)$ defined by $g \mapsto g \circ \iota$ is a bijection, and this restricts to a bijection $\mathrm{Epi}(\widehat{\Gamma}, Q) \to \mathrm{Epi}(\Gamma, Q)$.*

We record the following corollary for later use.

Corollary 4.2 *If Γ_1 is finitely-generated and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then*

$$|\mathrm{Hom}(\Gamma_1, Q)| = |\mathrm{Hom}(\Gamma_2, Q)|$$

for every finite group Q.

4.4. The first Betti number of a finitely generated group is

$$b_1(\Gamma) = \dim_{\mathbf{Q}} [(\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbf{Z}} \mathbf{Q}].$$

Given any prime p, one can detect $b_1(\Gamma)$ in the p-group quotients of Γ, since it is the *greatest integer b such that Γ surjects $(\mathbf{Z}/p^k\mathbf{Z})^b$ for every $k \in \mathbf{N}$*. We exploit this observation as follows:

Lemma 4.3 *Let Λ and Γ be finitely generated groups. If Λ is isomorphic to a dense subgroup of $\widehat{\Gamma}$, then $b_1(\Lambda) \geq b_1(\Gamma)$.*

Proof For every finite group A, each epimorphism $\widehat{\Gamma} \to A$ will restrict to an epimorphism on both Γ and Λ (since by density Λ cannot be contained in a proper closed subgroup). But the resulting map $\mathrm{Epi}(\widehat{\Gamma}, A) \to \mathrm{Epi}(\Lambda, A)$ need not be surjective, in contrast to Lemma 4.1. Thus if Γ surjects $(\mathbf{Z}/p^k\mathbf{Z})^b$ then so does Λ (but perhaps not vice versa). □

4.5. We now discuss the profinite topology on the discrete group Γ, its subgroups and the correspondence between the subgroup structure of Γ and $\widehat{\Gamma}$. We begin by recalling the profinite topology on Γ. This is the topology on Γ in which a base for the open sets is the set of all cosets of normal subgroups of finite index in Γ. Now given a tower $\mathcal{T}$ of finite index normal subgroups of Γ:

$$\Gamma > N_1 > N_2 > \ldots > N_k > \ldots$$

with $\bigcap N_k = 1$, this can be used to define an inverse system and thereby determines a completion of $\widehat{\Gamma}_{\mathcal{T}}$ (in which Γ will inject). Now if the inverse system determined by $\mathcal{T}$ is cofinal (recall §4.1) then the natural homomorphism $\widehat{\Gamma} \to \widehat{\Gamma}_{\mathcal{T}}$ is an isomorphism. That is to say $\mathcal{T}$ determines the full profinite topology of Γ.

The following is important in connecting the discrete and profinite worlds (see [56] 3.2.2, where here we use [50] to replace "open" by "finite index").

Notation Given a subset X of a profinite group G, we write $\overline{X}$ to denote the closure of X in G.

Proposition 4.4 *If Γ is a finitely generated residually finite group, then there is a one-to-one correspondence between the set $\mathcal{X}$ of subgroups of Γ that are open in the profinite topology on Γ, and the set $\mathcal{Y}$ of all finite index subgroups of $\widehat{\Gamma}$.*

Identifying Γ with its image in the completion, this correspondence is given by:

- *For $H \in \mathcal{X}$, $H \mapsto \overline{H}$.*

- *For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.*

If $H, K \in \mathcal{X}$ and $K < H$ then $[H : K] = [\overline{H} : \overline{K}]$. Moreover, $K \lhd H$ if and only if $\overline{K} \lhd \overline{H}$, and $\overline{H}/\overline{K} \cong H/K$.

The following corollary of this correspondence will be useful in what follows.

Corollary 4.5 *Let Γ be a finitely-generated group, and for each $d \in \mathbf{N}$, let M_d denote the intersection of all normal subgroups of index at most d in Γ. Then the closure $\overline{M}_d$ of M_d in $\widehat{\Gamma}$ is the intersection of all normal subgroups of index at most d in $\widehat{\Gamma}$, and hence $\bigcap_{d\in\mathbb{N}} \overline{M}_d = 1$.*

Proof If N_1 and N_2 are the kernels of epimorphisms from Γ to finite groups Q_1 and Q_2, then $\overline{N_1 \cap N_2}$ is the kernel of the extension of $\Gamma \to Q_1 \times Q_2$ to $\widehat{\Gamma}$, while $\overline{N}_1 \times \overline{N}_2$ is the kernel of the map $\widehat{\Gamma} \to Q_1 \times Q_2$ that one gets by extending each of $\Gamma \to Q_i$ and then taking the direct product. The uniqueness of extensions tells us that these maps coincide, and hence $\overline{N_1 \cap N_2} = \overline{N}_1 \cap \overline{N}_2$. The claims follow from repeated application of this observation. □

If now $H < \Gamma$, the profinite topology on Γ determines some pro topology on H and therefore some completion of H. To understand what happens in certain cases that will be of interest to us, we recall the following. Since we are assuming that Γ is residually finite, H injects into $\widehat{\Gamma}$ and determines a subgroup $\overline{H}$. Hence there is a natural epimorphism $\widehat{H} \to \overline{H}$. This need not be injective. For this to be injective (i.e. the full profinite topology is induced on H) we require the following to hold:

> *For every subgroup H_1 of finite index in H, there exists a finite index subgroup $\Gamma_1 < \Gamma$ such that $\Gamma_1 \cap H < H_1$.*

There are some important cases for which injectivity can be arranged. Suppose that Γ is a group and H a subgroup of Γ, then Γ is called H-separable if for every $g \in G \setminus H$, there is a subgroup K of finite index in Γ such that $H \subset K$ but $g \notin K$; equivalently, the intersection of all finite index subgroups in Γ containing H is precisely H. The group Γ is called *LERF* (or *subgroup separable*) if it is H-separable for every finitely-generated subgroup H, or equivalently, if every finitely-generated subgroup is a closed subset in the profinite topology.

It is important to note that even if the subgroup H of Γ is separable, it need not be the case that the profinite topology on Γ induces the full profinite topology on H. Stronger separability properties do suffice, however, as we now indicate.

Lemma 4.6 *Let Γ be a finitely-generated group, and H a finitely-generated subgroup of Γ. Suppose that Γ is H_1-separable for every finite index subgroup H_1 in H. Then the profinite topology on Γ induces the full profinite topology on H; that is, the natural map $\widehat{H} \to \overline{H}$ is an isomorphism.*

Proof Since Γ is H_1 separable, the intersection of all subgroups of finite index in Γ containing H_1 is H_1 itself. From this it easily follows that there exists $\Gamma_1 < \Gamma$ of finite index, so that $\Gamma_1 \cap H = H_1$. The lemma follows from the discussion above. □

Subgroups of finite index obviously satisfy the conditions of Lemma 4.6, and if Γ is LERF, the conditions of Lemma 4.6 are also satisfied. Hence we deduce the following.

Corollary 4.7 (1) *If Γ is residually finite and H is a finite-index subgroup of Γ, then the natural map from $\widehat{H}$ to $\overline{H}$ is an isomorphism.*

(2) *If Γ is LERF and H is a finitely generated subgroup of Γ, then the natural map from $\widehat{H}$ to $\overline{H}$ is an isomorphism.*

Another case of what the profinite topology does on a subgroup that will be of interest to us is the following. Let Γ be a residually finite group that is the fundamental group of a graph of groups. Let the edge groups be denoted by G_e and the vertex groups by G_v. The profinite topology on Γ is said to be *efficient* if it induces the full profinite topology on G_v and G_e for all vertex and edge groups, and G_v and G_e are closed in the profinite topology on Γ. The main example we will make use of is the following which is well-known:

Lemma 4.8 *Suppose that Γ is a free product of finitely many residually finite groups $G_1, \ldots, G_n$. Then the profinite topology on Γ is efficient.*

Proof Since Γ is residually finite, the trivial group is closed in the profinite topology. To see that each G_i is closed in the profinite topology we prove that Γ is G_i-separable. To that end let G denote one of the G_i, and let $g \in \Gamma \setminus G$. Since $g \notin G$, the normal form for g contains at least one element $a_k \in G_k \neq G$. Since G_k is residually finite there is a finite quotient A of G_k for which the image of a_k is non-trivial. Using the projection homomorphism $G_1 * \ldots * G_n \to G_k \to A$ defines a homomorphism for which the image of G is trivial but the image of g is not. This proves the vertex groups are closed.

To see that the full profinite topology is induced on each G_i, we need to show that for each G_i, $i = 1, \ldots, n$, the following condition holds (recall the condition for injectivity given above). For every subgroup H of finite index in G_i, there exists a finite index subgroup $H_i < \Gamma$ such that $H_i \cap G_i < H$. Let G denote one of the G_i's and assume that $H < G$ is a finite index subgroup. We can also assume that H is a normal subgroup. Then using the projection homomorphism $\Gamma = G_1 * \ldots * G_n \to G/H$ whose kernel K defines a finite index of subgroup of Γ with $K \cap G = H$ as required. $\square$

Note that in the situation of Lemma 4.8, it also follows that $\widehat{\Gamma} \cong \widehat{G}_1 \amalg \widehat{G}_2 \ldots \amalg \widehat{G}_n$ where $\amalg$ indicates the profinite amalgamated product. We refer the reader to [56] Chapter 9 for more on this.

4.6. We now prove one of the key results that we make use of. This is basically proved in [25] (see also [56] pp. 88–89), the mild difference here, is that we employ [50] to replace *topological isomorphism* with *isomorphism*.

Theorem 4.9 *Suppose that Γ_1 and Γ_2 are finitely-generated abstract groups. Then $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic if and only if $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

Proof If $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic then the discussion following the correspondence provided by Proposition 4.4 shows that $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.

For the converse, we argue as follows. For each $n \in \mathbf{N}$ let

$$U_n = \bigcap\{U : U \text{ is a normal subgroup of } \Gamma_1 \text{ with } [\Gamma_1 : U] \leq n\}, \quad \text{and}$$
$$V_n = \bigcap\{V : V \text{ is a normal subgroup of } \Gamma_2 \text{ with } [\Gamma_2 : V] \leq n\}.$$

Then $\Gamma_1/U_n \in \mathcal{C}(\Gamma_1)$ and $\Gamma_2/V_n \in \mathcal{C}(\Gamma_2)$. Hence there exists a normal subgroup $K < \Gamma_1$ so that $\Gamma_1/K \cong \Gamma_2/V_n$. Now it follows that K is an intersection of normal subgroups of index $\leq n$, and so $U_n < K$. Hence $|\Gamma_2/V_n| = |\Gamma_1/K| \leq |\Gamma_1/U_n|$. On reversing the roles of Γ_1 and Γ_2 reverses this inequality from which it follows that $\Gamma_2/V_n \cong \Gamma_1/U_n$.

Now for each such n, let A_n denote the set of all isomorphisms Γ_1/U_n onto Γ_2/V_n. For each n this is a finite non-empty set with the property that for $m \leq n$ and $\alpha \in A_n$, then α induces a unique homomorphism $f_{nm}(\alpha) : \Gamma_1/U_m \to \Gamma_2/V_m$ such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma_1/U_n & \longrightarrow & \Gamma_1/U_m \\ \alpha\big\downarrow & & \big\downarrow f_{nm}(\alpha) \\ \Gamma_2/V_n & \longrightarrow & \Gamma_2/V_m \end{array}$$

It follows that $\{A_n, f_{nm}\}$ is an inverse system of (non-empty) finite sets, and so the inverse limit $\varprojlim A_n$ exists and defines an isomorphism of the inverse systems $\varprojlim \Gamma_1/U_n$ and $\varprojlim \Gamma_2/V_n$. Also note that since U_n and V_n are co-final, the discussion in §4.5 shows that they induce the full profinite topology on Γ_1 and Γ_2 respectively and so we have:

$$\widehat{\Gamma}_1 \cong \varprojlim \Gamma_1/U_n \cong \varprojlim \Gamma_2/V_n \cong \widehat{\Gamma}_2$$

as required. □

Thus statements about $\mathcal{C}(\Gamma)$ and $\mathcal{G}(\Gamma)$ can now be rephrased in terms of the profinite completion. For example,

$$\mathcal{G}(\Gamma) = \{\Delta : \widehat{\Delta} \cong \widehat{\Gamma}\}.$$

4.7. We now give some immediate applications of Theorem 4.9 and the previous discussion in the context of the motivating questions.

Lemma 4.10 *Let $\phi : \Gamma_1 \to \Gamma_2$ be an epimorphism of finitely-generated groups. If Γ_1 is residually finite and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then ϕ is an isomorphism.*

Proof Let $k \in \ker \phi$. If k were non-trivial, then since Γ_1 is residually finite, there would be a finite group Q and an epimorphism $f : \Gamma_1 \to Q$ such that $f(k) \neq 1$. This map f does not lie in the image of the injection $\mathrm{Hom}(\Gamma_2, Q) \hookrightarrow \mathrm{Hom}(\Gamma_1, Q)$ defined by $g \mapsto g \circ \phi$. Thus $|\mathrm{Hom}(\Gamma_1, Q)| > |\mathrm{Hom}(\Gamma_2, Q)|$, contradicting Corollary 4.2. □

Definition 4.11 The *rank* $d(\Gamma)$ of a finitely-generated group Γ is the least integer k such that Γ has a generating set of cardinality k. The *rank* $\widehat{d}(G)$ of a profinite group G is the least integer k for which there is a subset $S \subset G$ with $k = |S|$ and $\langle S \rangle$ is dense in G.

If Γ_1 is assumed to be a finitely generated free group of rank r and Γ_2 a finitely generated group with $d(\Gamma_2) = r$ and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then it follows immediately from Lemma 4.10 that Γ_2 is isomorphic to a free group of rank r (using the natural epimorphism $\Gamma_1 \to \Gamma_2$).

Indeed, one can refine this line of argument as follows. In the following proposition, we do not assume that Γ is residually finite.

Proposition 4.12 *Let Γ be a finitely-generated group and let F_n be a free group. If Γ has a finite quotient Q such that $d(\Gamma) = d(Q)$, and $\widehat{\Gamma} \cong \widehat{F}_n$, then $\Gamma \cong F_n$.*

Proof First $\widehat{\Gamma} \cong \widehat{F}_n$, so Q is a quotient of F_n. Hence $n \geq d(Q)$. But $d(Q) = d(\Gamma)$ and for every integer $s \geq d(\Gamma)$ there exists an epimorphism $F_s \to \Gamma$. Thus we obtain an epimorphism $F_n \to \Gamma$, and application of the preceding lemma completes the proof. □

Corollary 4.13 *Let Γ be a finitely-generated group. If Γ and its abelianisation have the same rank, then $\widehat{\Gamma} \cong \widehat{F}_n$ if and only if $\Gamma \cong F_n$.*

Proof Every finitely-generated abelian group A has a finite quotient of rank $d(A)$. □

As an application of Corollary 4.13 we give a quick proof that that free groups and surface groups are distinguished by their finite quotients. For if Γ is a genus $g \geq 1$ surface group, then Γ and its abelianization have rank $2g$. Corollary 4.13 then precludes such a group having the same profinite completion as a free group.

Another application is the following. Another natural generalization of free groups are right angled Artin groups. Let K be a finite simplicial graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E \subset V \times V$. Then the *right angled Artin group* (or $RAAG$) associated with K is the group $A(K)$ given by the following presentation:

$$A(K) = \langle\, v_1, \ldots, v_n \mid [v_i, v_j] = 1 \text{ for all } i, j \text{ such that } \{v_i, v_j\} \in E \,\rangle.$$

For example, if K is a graph with n vertices and no edges, then $A(K)$ is the free group of rank n, while if K is the complete graph on n vertices, then $A(K)$ is the free abelian group $\mathbf{Z}^n$ of rank n.

If the group Γ has a presentation of the form $\langle\, A \mid R \,\rangle$ where A is finite and all of the relators $r \in R$ lie in the commutator subgroup of the free group $F(A)$, then both Γ and its abelianisation (which is free abelian) have rank $|A|$. The standard presentations of RAAGs have this form.

Proposition 4.14 *If Γ is a right-angled Artin group that is not free, then there exists no free group F such that $\widehat{F} \cong \widehat{\Gamma}$.*

4.8. We shall also consider other pro-completions, and we briefly recall these. The *pro-(finite nilpotent)* completion, denoted $\widehat{\Gamma}_{\mathrm{fn}}$, is the inverse limit of the finite nilpotent quotients of Γ. Given a prime p, the the *pro-p completion* $\widehat{\Gamma}_p$ is the inverse limit of the finite p-group quotients of Γ. As above we have natural homomorphisms $\Gamma \to \widehat{\Gamma}_{\mathrm{fn}}$

and $\Gamma \to \widehat{\Gamma}_p$ and these are injections if and only if Γ is residually nilpotent in the first case and residually p in the second.

Note that in this language, two finitely generated residually nilpotent groups with the same nilpotent genus have isomorphic pro-(finite nilpotent) completions. This can proved in a similar manner as Proposition 4.4 using only the finite nilpotent quotients. Note that it is proved in [6] (before the general case of [50]) that for a finitely generated group Γ, *every* subgroup of finite index in $\widehat{\Gamma}_{\rm fn}$ is open. Moreover, finitely generated groups in the same nilpotent genus also have isomorphic pro-p completions for all primes p.

5 Grothendieck Pairs and Grothendieck Rigidity

A particular case of when discrete groups have isomorphic profinite completions is the following (which goes back to Grothendieck [28]).

5.1. Let Γ be a residually finite group and let $u : P \hookrightarrow \Gamma$ be the inclusion of a subgroup P. Then $(\Gamma, P)_u$ is called a *Grothendieck Pair* if the induced homomorphism $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism but u is not. (When no confusion is likely to arise, it is usual to write (Γ, P) rather than $(\Gamma, P)_u$.) Grothendieck [28] asked about the existence of such pairs of finitely presented groups and the first such pairs were constructed by Bridson and Grunewald in [15]. The analogous problem for finitely generated groups had been settled earlier by Platonov and Tavgen [54]. Both constructions rely on versions of the following result (cf. [54], [15] Theorem 5.2 and [13]).

We remind the reader that the *fibre product* $P < \Gamma \times \Gamma$ associated to an epimorphism of groups $p : \Gamma \to Q$ is the subgroup $P = \{(x, y) : p(x) = p(y)\}$.

Proposition 5.1 *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups with Γ finitely generated and let P be the associated fibre product. Suppose that $Q \neq 1$ is finitely presented, has no proper subgroups of finite index, and $H_2(Q, \mathbf{Z}) = 0$. Then*

(1) *$(\Gamma \times \Gamma, P)$ is a Grothendieck Pair;*

(2) *if N is finitely generated then (Γ, N) is a Grothendieck Pair.*

More recently in [14], examples of Grothendieck Pairs were constructed so as to provide the first examples of finitely-presented, residually finite groups Γ that contain an infinite sequence of non-isomorphic finitely presented subgroups P_n so that the inclusion maps $u_n : P_n \hookrightarrow \Gamma$ induce isomorphisms of profinite completions. In particular, this provides examples of finitely presented groups Γ for which $\mathcal{G}(\Gamma)$ is infinite.

5.2. There are many classes of groups Γ that can never have a subgroup P for which (Γ, P) is a Grothendieck Pair; as in [40], we call such groups *Grothendieck Rigid*.

Before proving the next theorem, we make a trivial remark that is quite helpful. Suppose that $H < \Gamma$ and Γ is H-separable, then (Γ, H) is not a Grothendieck Pair. The reason for this is that being separable implies that H is contained in (infinitely many) proper subgroups of Γ of finite index. In particular $\overline{H} < \widehat{\Gamma}$ is contained in proper subgroups of finite index in $\widehat{\Gamma}$. On the other hand if (Γ, H) is a Grothendieck Pair, H is dense in $\widehat{\Gamma}$ and so cannot be contained in a closed subgroup (of finite index)

of $\widehat{\Gamma}$. With this remark in place, we prove our next result. Recall that a group Γ is called *residually free* if for every non-trivial element $g \in \Gamma$ there is a homomorphism ϕ_g from Γ to a free group such that $\phi_g(g) \neq 1$, and Γ is *fully residually free* if for every finite subset $X \subseteq \Gamma$ there is a homomorphism from Γ to a free group that restricts to an injection on X.

Theorem 5.2 *Let Γ be a finitely generated group isomorphic to either: a Fuchsian group, a Kleinian group, the fundamental group of a geometric 3-manifold, a fully residually free group. Then Γ is Grothendieck Rigid.*

Proof This follows immediately from the discussion above, and the fact that such groups are known to be LERF. For Fuchsian groups see [57], for Kleinian groups this follows from [2] and [62] and for fully residually free groups [60]. If M is a geometric 3-manifold, then the case when M is hyperbolic follows from the remark above, and when M is a Seifert fibered space see [57]. For those modelled on SOL geometry, separability of subgroups can be established directly and the result follows. □

Remark The case of finite co-volume Kleinian groups was proved in [40] without using the LERF assumption. Instead, character variety techniques were employed. In §8.2 we will establish Grothendieck Rigidity for prime 3-manifolds that are not geometric.

6 L^2-Betti numbers and profinite completion

Proposition 3.2 established that the first Betti number of a group is a profinite invariant. The goal of this section is to extend this to the first L^2-Betti number, and to give some applications of this.

We refer the reader to Lück's paper [47] for a comprehensive introduction to L^2-Betti numbers. For our purposes, it is best to view these invariants not in terms of their original (more analytic) definition, but instead as asymptotic invariants of towers of finite-index subgroups. This is made possible by the Lück's Approximation Theorem [46]:

Theorem 6.1 *Let Γ be a finitely presented group, and let $\Gamma = \Gamma_1 > \Gamma_2 > \ldots > \Gamma_m > \ldots$ be a sequence of finite-index subgroups that are normal in Γ and intersect in the identity. Then for all $p \geq 0$, the p-th L^2-Betti number of Γ is given by the formula*

$$b_p^{(2)}(\Gamma) = \lim_{m\to\infty} \frac{b_p(\Gamma_m)}{[\Gamma : \Gamma_m]}.$$

An important point to note is that this limit does not depend on the tower, and hence is an invariant of Γ. We will mostly be interested in $b_1^{(2)}$.

Example 6.2 Let F be a free group of rank r. Euler characteristic tells us that a subgroup of index d in F is free of rank $d(r-1)+1$, so by Lück's Theorem $b_1^{(2)}(F_r) = r-1$. A similar calculation shows that if Σ is the fundamental group of a closed surface of genus g, then $b_1^{(2)}(\Sigma) = 2g-2$.

Proposition 6.3 *Let Λ and Γ be finitely presented residually finite groups and suppose that Λ is a dense subgroup of $\widehat{\Gamma}$. Then $b_1^{(2)}(\Gamma) \leq b_1^{(2)}(\Lambda)$.*

Proof For each positive integer d let M_d be the intersection of all normal subgroups of index at most d in Γ, and let $L_d = \Lambda \cap \overline{M}_d$ in $\widehat{\Gamma}$. We saw in Corollary 4.5 that $\bigcap_d \overline{M}_d = 1$, and so $\bigcap_d L_d = 1$. Since Λ and Γ are both dense in $\widehat{\Gamma}$, the restriction of $\widehat{\Gamma} \to \widehat{\Gamma}/\overline{M}_d$ to each of these subgroups is surjective, and hence

$$[\Lambda : L_d] = [\widehat{\Gamma} : \overline{M}_d] = [\Gamma : M_d].$$

Now L_d is dense in $\overline{M}_d$, while $\widehat{M_d} = \overline{M}_d$, so Lemma 4.3 implies that $b_1(L_d) \geq b_1(M_d)$, and then we can use the towers (L_d) in Λ and (M_d) in Γ to compare L^2-Betti numbers and find

$$b_1^{(2)}(\Gamma) = \lim_{d\to\infty} \frac{b_1(M_d)}{[\Gamma : M_d]} \leq \lim_{d\to\infty} \frac{b_1(L_d)}{[\Lambda : L_d]} = b_1^{(2)}(\Lambda),$$

by Lück's approximation theorem. □

This has the following important consequence:

Corollary 6.4 *Let Γ_1 and Γ_2 be finitely-presented residually finite groups. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $b_1^{(2)}(\Gamma_1) = b_1^{(2)}(\Gamma_2)$.*

If one assumes only that the group Γ is *finitely generated*, then one does not know if the above limit exists, and when it does exist one does not know if it is independent of the chosen tower of subgroups. However, a weaker form of Lück's approximation theorem for $b_1^{(2)}$ was established for finitely generated groups by Lück and Osin [48].

Theorem 6.5 *If Γ is a finitely generated residually finite group and (N_m) is a sequence of finite-index normal subgroups with $\bigcap_m N_m = 1$, then*

$$\limsup_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} \leq b_1^{(2)}(\Gamma).$$

6.1. We now give some applications of Proposition 6.3 in the context of Question 1 (and the analogous questions for Fuchsian groups). First we generalize the calculation in Example 6.2.

Proposition 6.6 *If Γ is a lattice in $\mathrm{PSL}(2,\mathbf{R})$ with rational Euler characteristic $\chi(\Gamma)$, then $b_1^{(2)}(\Gamma) = -\chi(\Gamma)$.*

Proof It follows from Lück's approximation theorem that if H is a subgroup of index index d in Γ (which is finitely-presented) then $b_1^{(2)}(H) = d\, b_1^{(2)}(\Gamma)$. Rational Euler characteristic is multiplicative in the same sense. Thus we may pass to a torsion-free subgroup of finite index in Γ, and assume that it is either a free group F_r of rank r, or the fundamental group Σ_g of a closed orientable surface of genus g. The free group case was dealt with above, and so we focus on the surface group case.

Thus if Γ_d is a subgroup of index d in Γ, then it is a surface group of genus $d(g-1)+1$. The first Betti number in this case is $2d(g-1)+1$ and so $b_1(\Gamma_d) = 2 - d\,\chi(\Gamma)$. Dividing by $d = |\Gamma : \Gamma_d|$ and taking the limit, we find $b_1^{(2)}(\Gamma) = -\chi(\Gamma)$. □

With this result and Proposition 6.3 we have the following. The only additional comment to make is that the assumption that the Fuchsian group Γ_1 is non-elementary implies it is not virtually abelian, and so $b_1^{(2)}(\Gamma_1) \neq 0$.

Corollary 6.7 *Let Γ_1 be a finitely generated non-elementary Fuchsian group, and Γ_2 a finitely presented residually finite group with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Then $b_1^{(2)}(\Gamma_2) = b_1^{(2)}(\Gamma_1) = -\chi(\Gamma_1) \neq 0$.*

Another standard result about free groups is that if F is a finitely generated free group of rank ≥ 2, then any finitely generated non-trivial normal subgroup of F has finite index (this also holds more generally for Fuchsian groups and limit groups, see [17] for the last statement). As a further corollary of Propositon 6.4 we prove the following.

Corollary 6.8 *Let Γ be a finitely presented residually finite group in the same genus as a finitely generated free group, and let $N < \Gamma$ be a non-trivial normal subgroup. If N is finitely generated, then Γ/N is finite.*

Proof Proposition 3.1 shows that the genus of the infinite cyclic group contains only itself, and so we can assume that Γ lies in the genus of a non-abelian free group. Thus, by Corollary 6.4, $b_1^{(2)}(\Gamma) \neq 0$. The proof is completed by making use of the following theorem of Gaboriau (see [27] Theorem 6.8):

Theorem 6.9 *Suppose that*

$$1 \to N \to \Gamma \to \Lambda \to 1$$

is an exact sequence of groups where N and Λ are infinite. If $b_1^{(2)}(N) < \infty$, then $b_1^{(2)}(\Gamma) = 0$.

□

Indeed, using Theorem 6.5, Corollary 6.8 can be proved under the assumption that Γ is a finitely generated residually finite group. In this case, the argument establishes that if Γ is in the same genus as a finitely generated free group F, then $b_1^{(2)}(\Gamma) \geq b_1^{(2)}(F)$ and we can still apply [27].

As remarked upon earlier, Question 1 is still unresolved, and in the light of this, Corollary 6.8 provides some information about the structural properties of a finitely generated group in the same genus as a free group. In §8.3, we point out some other properties that occur assuming that a group Γ is in the same genus as a finitely generated free group.

Remark Unlike the case of surface groups, if M is a closed 3-manifold, then typically $b_1^{(2)}(\pi_1(M)) = 0$. More precisely, we have the following from [44]. Let $M = M_1 \# M_2 \# \ldots \# M_r$ be the connect sum of closed (connected) orientable prime 3-manifolds and that $\pi_1(M)$ is infinite. Then

$$b_1^{(2)}(\pi_1(M)) = (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|},$$

where in the summation, if $\pi_1(M_j)$ is infinite, the term in the sum is understood to be zero.

6.2. Corollary 6.4 establishes that $b_1^{(2)}$ is an invariant for finitely presented groups in the same genus. A natural question arises as to whether anything can be said about the higher L^2-Betti numbers. Using the knowledge of L^2-Betti numbers of locally symmetric spaces (see [21]), it follows that the examples given §3.4 will have all $b_p^{(2)}$ equal. On the other hand, using [4] examples can be constructed which *do not* have all $b_p^{(2)}$ being equal. Further details will appear elsewhere.

7 Goodness

In this section we discuss how cohomology of profinite groups can be used to inform about Questions 1–5.

7.1. We begin by recalling the definition of the *continuous cohomology* of profinite groups (also known as *Galois cohomology*). We refer the reader to [59] and [56, Chapter 6] for details about the cohomology of profinite groups.

Let G be a profinite group, M a discrete G-module (i.e., an abelian group M equipped with the discrete topology on which G acts continuously) and let $C^n(G, M)$ be the set of all continous maps $G^n \to M$. One defines the coboundary operator $d : C^n(G, M) \to C^{n+1}(G, M)$ in the usual way thereby defining a complex $C^*(G, M)$ whose cohomology groups $H^q(G; M)$ are called the continuous cohomology groups of G with coefficients in M.

Note that $H^0(G; M) = \{x \in M : gx = x \ \forall \ g \in G\} = M^G$ is the subgroup of elements of M invariant under the action of G, $H^1(G; M)$ is the group of classes of continuous crossed homomorphisms of G into M and $H^2(G; M)$ is in one-to-one correspondence with the (equivalence classes of) extensions of M by G.

7.2. Now let Γ be a finitely generated group. Following Serre [59], we say that a group Γ is *good* if for all $q \geq 0$ and for every finite Γ-module M, the homomorphism of cohomology groups

$$H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$.

Example 7.1 Finitely generated free groups are good.

To see this we argue as follows. As is pointed out by Serre ([59] p. 15), for any (finitely generated) discrete group Γ, one always has isomorphisms $H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$ for $q = 0, 1$. Briefly, using the description of H^0 given above (and the discrete setting), isomorphism for H^0 follows using denseness of Γ in $\widehat{\Gamma}$ and discreteness of M. For H^1, this follows using the description of H^1 as crossed homomorphisms.

If Γ is now a finitely generated free group, since $H^2(\widehat{G}; M)$ is in one-to-one correspondence with the (equivalence classes of) extensions of M by $\widehat{\Gamma}$, it follows that $H^2(\widehat{\Gamma}; M) = 0$ (briefly, like the case of the discrete free group there are no interesting extensions).

The higher cohomology groups $H^q(\widehat{\Gamma}; M)$ $(q \geq 3)$ can also be checked to be zero. For example, since $H^q(\Gamma; M) = 0$ for all $q \geq 2$, the induced map $H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$ is surjective for all $q \geq 2$, and it now follows from a lemma of Serre [59] (see Ex 1 Chapter 2, and also Lemma 2.1 of [41]) that $H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$ is injective for all $q \geq 2$. We also refer the reader to the discussion below on cohomological dimension for another approach.

Goodness is hard to establish in general. One can, however, establish goodness for a group Γ that is LERF if one has a well-controlled splitting of the group as a graph of groups [30]. In addition, a useful criterion for goodness is provided by the next lemma due to Serre (see [59, Chapter 1, Section 2.6])

Lemma 7.2 *The group Γ is good if there is a short exact sequence*

$$1 \to N \to \Gamma \to H \to 1,$$

such that H and N are good, N is finitely-generated, and the cohomology group $H^q(N, M)$ is finite for every q and every finite Γ-module M.

We summarize what we will need from this discussion.

Theorem 7.3 *The following classes of groups are good.*

- *Finitely generated Fuchsian groups.*
- *The fundamental groups of compact 3-manifolds.*
- *Fully residually free groups.*
- *Right angled Artin groups.*

Proof The first and third parts are proved in [30] using LERF and well-controlled splittings of the group, and the fourth is proved in [41]. The second was proved by Cavendish in his PhD thesis [23]. We will sketch the proof when M is closed.

Note first that by [30] free products of residually finite good groups are good, so it suffices to establish goodness for prime 3-manifolds. As is shown in [30] goodness is preserved by commensurability, and so finite groups are clearly good. Thus it remains to establish goodness for prime 3-manifolds with infinite fundamental group. For geometric closed 3-manifolds, goodness will follow immediately from Lemma 7.2 (using the first part of the theorem) when $\Gamma = \pi_1(M)$ and M is a Seifert fibered space or has SOL geometry. For hyperbolic 3-manifolds the work of Agol [2] and Wise [62] shows that any finite volume hyperbolic 3-manifold has a finite cover that fibers over the circle, and once again by Lemma 7.2 (and the first part of the theorem) we deduce

goodness. For manifolds with a non-trivial JSJ decomposition, goodness is proved in [61]. □

7.3. Let G be a profinite group. Then the *p-cohomological dimension* of G is the least integer n such that for every finite (discrete) G-module M and for every $q > n$, the p-primary component of $H^q(G; M)$ is zero, and this is denoted by $\mathrm{cd}_p(G)$. The cohomological dimension of G is defined as the supremum of $\mathrm{cd}_p(G)$ over all primes p, and this is denoted by $\mathrm{cd}(G)$.

We also retain the standard notation $\mathrm{cd}(\Gamma)$ for the cohomological dimension (over $\mathbf{Z}$) of a *discrete* group Γ. A basic connection between the discrete and profinite versions is given by

Lemma 7.4 *Let Γ be a discrete group that is good. If $\mathrm{cd}(\Gamma) \leq n$, then $\mathrm{cd}(\widehat{\Gamma}) \leq n$.*

Proof If $\mathrm{cd}(\Gamma) \leq n$ then $H^q(\Gamma, M) = 0$ for every Γ-module M and every $q > n$. By goodness this transfers to the profinite setting in the context of finite modules. □

Discrete groups of finite cohomological dimension (over $\mathbf{Z}$) are torsion-free. In connection with goodness, we are interested in conditions that allow one to deduce that $\widehat{\Gamma}$ is also torsion-free. For this we need the following result that mirrors the behavior of cohomological dimension for discrete groups (see [59, Chapter 1 §3.3]).

Proposition 7.5 *Let p be a prime, let G be a profinite group, and H a closed subgroup of G. Then $\mathrm{cd}_p(H) \leq \mathrm{cd}_p(G)$.*

This quickly yields the following that we shall use later.

Corollary 7.6 *Suppose that Γ is a residually finite, good group of finite cohomological dimension over $\mathbf{Z}$. Then $\hat{\Gamma}$ is torsion-free.*

Proof If $\widehat{\Gamma}$ were not torsion-free, then it would have an element x of prime order, say q. Since $\langle x \rangle$ is a closed subgroup of $\widehat{\Gamma}$, Proposition 7.5 tells us that $\mathrm{cd}_p(\langle x \rangle) \leq \mathrm{cd}_p(\widehat{\Gamma})$ for all primes p. But $H^{2k}(\langle x \rangle; \mathbf{F}_q) \neq 0$ for all $k > 0$, so $\mathrm{cd}_q(\langle x \rangle)$ and $\mathrm{cd}_q(\widehat{\Gamma})$ are infinite. Since Γ is good and has finite cohomological dimension over $\mathbf{Z}$, we obtain a contradiction from Lemma 7.4. □

Note that this can be used to exhibit linear groups that are not good. For example, in [45], it is shown that there are torsion-free subgroups $\Gamma < \mathrm{SL}(n, \mathbf{Z})$ ($n \geq 3$) of finite index, for which $\widehat{\Gamma}$ contains torsion of all possible orders. As a corollary of this we have:

Corollary 7.7 *For all $n \geq 3$, any subgroup of $\mathrm{SL}(n, \mathbf{R})$ commensurable with $\mathrm{SL}(n, \mathbf{Z})$ is not good.*

7.4. When the closed subgroup is a p-Sylow subgroup G_p (i.e., a maximal closed pro-p subgroup of G) then we have the following special case of Proposition 7.5 (see [56] §7.3). Note that cohomology theory of pro-p groups is easier to understand than general profinite groups, and so the lemma is quite helpful in connection with computing cohomology of profinite groups.

Lemma 7.8 *Let G_p be a p-Sylow subgroup of G. Then:*

- $\mathrm{cd}_p(G) = \mathrm{cd}_p(G_p) = \mathrm{cd}(G_p)$.
- $\mathrm{cd}(G) = 0$ *if and only if* $G = 1$.
- $\mathrm{cd}_p(G) = 0$ *if and only* $G_p = 1$.

Example 7.9 Let F be a finitely generated free group. Since a p-Sylow subgroup of $\widehat{F}$ is $\mathbf{Z}_p$, Lemma 7.8 gives an efficient way to establish that $\mathrm{cd}(\widehat{F}) = 1$.

7.5. In this subsection we point out how goodness (in fact a weaker property suffices) provides a remarkable condition to establish residual finiteness of extensions. First suppose that we have an extension:

$$1 \to N \to E \to \Gamma \to 1.$$

Using right exactness of the profinite completion (see [56] Proposition 3.2.5), this short exact sequence always determines a sequence:

$$\widehat{N} \to \widehat{E} \to \widehat{\Gamma} \to 1.$$

To ensure that the induced homorphism $\widehat{N} \to \widehat{E}$ is injective is simply again the statement that the full profinite topology is induced on N. As was noticed by Serre [59], this is guaranteed by goodness. Indeed the following is true, the proof of which we discuss below (the proof is sketched in [59] and see also [30] and [41]).

Proposition 7.10 *The following are equivalent for a group Γ.*

- *For any finite Γ-module M, the induced map $H^2(\widehat{\Gamma}; M) \to H^2(\Gamma; M)$ is an isomorphism;*
- *For every group extension $1 \to N \to E \to \Gamma \to 1$ with N finitely generated, the map $\widehat{N} \to \widehat{E}$ is injective.*

Before discussing this we deduce the following.

Corollary 7.11 *Suppose that Γ is residually finite and for any finite Γ-module M, the induced map $H^2(\widehat{\Gamma}; M) \to H^2(\Gamma; M)$ is an isomorphism. Then any extension E (as above) by a finitely generated residually finite group N is residually finite.*

Groups as in Corollary 7.11 are called *highly residually finite* in [41], and *super residually finite* in [22].

Proof By Proposition 7.10, and referring to the diagram below, we have exact sequences with vertical homomorphisms i_N and i_Γ being injective by residual finiteness. Now the squares commute, and so a 5-Lemma argument implies that i_E is injective; i.e., E is residually finite.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow i_N & & \downarrow i_E & & \downarrow i_\Gamma & & \\ 1 & \longrightarrow & \widehat{N} & \longrightarrow & \widehat{E} & \longrightarrow & \widehat{\Gamma} & \longrightarrow & 1 \end{array}$$

□

Proof We discuss the "if" direction below, and refer the reader to [41] for the "only if". We will show that it suffices to prove the result with N finite. For then the case of N finite is dealt with by Proposition 6.1 of [30].

Thus assume that N is finitely generated and J a finite index subgroup of N. Recall that from §4.5 we need to show that there exists a finite index subgroup $E_1 < E$ such that $E_1 \cap N < J$.

To that end, since N is finitely generated we can find a characteristic subgroup $H < J$ of finite index in N that is normal in E. Thus we have:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \pi & & & & \\ 1 & \longrightarrow & N/H & \longrightarrow & E/H & \longrightarrow & \Gamma & \longrightarrow & 1 \end{array}$$

Assuming that the result holds for the case of N finite we can appy this to N/H. That is to say we can find $E_0' < E/H$ such that $E_0' \cap (N/H) = 1$. Set $E_0 = \pi^{-1}(E_0')$, then $E_0 \cap N < H < J$ as required. □

Given Corollary 7.11 and Theorem 7.3 we have:

Corollary 7.12 *Let Γ be a group as in Theorem 7.3. Then Γ is highly residually finite.*

Examples of groups that are not highly residually finite are $\mathrm{SL}(3, \mathbf{Z})$ (see [33]) and $\mathrm{Sp}(2g, \mathbf{Z})$ ([24]). In particular in [24] lattices in a connected Lie group are constructed that are not residually finite. These arise as extensions of $\mathrm{Sp}(2g, \mathbf{Z})$.

7.6. We now return to Question 1, and in particular deduce some consequences about a group Γ in the same genus as a finitely generated free group. To that end, the following simple observation will prove useful.

Corollary 7.13 *Let Γ_1 and Γ_2 be finitely-generated (abstract) residually finite groups with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Assume that Γ_1 is good and $\mathrm{cd}(\Gamma_1) = n < \infty$. Furthermore, assume that H is a good subgroup of Γ_2 for which the natural mapping $\widehat{H} \to \widehat{\Gamma}_2$ is injective. Then $H^q(H; \mathbf{F}_p) = 0$ for all $q > n$.*

Proof If $H^q(H; \mathbf{F}_p)$ were non-zero for some $q > n$, then by goodness we would have $H^q(\widehat{H}; \mathbf{F}_p) \neq 0$, so $\mathrm{cd}_p(\widehat{H}) \geq q > n$. Now $\widehat{H} \to \widehat{\Gamma}_2$ is injective and so $\widehat{H} \cong \overline{H}$. Hence $\widehat{\Gamma}_1$ contains a closed subgroup of p-cohomological dimension greater than n, a contradiction. □

Corollary 7.14 *If Γ contains a surface group S, and $\widehat{S} \to \widehat{\Gamma}$ is injective, then $\widehat{\Gamma}$ is not isomorphic to the profinite completion of any free group.*

In particular, this also shows the following:

Corollary 7.15 *If L is a non-abelian free group, then $\widehat{L}$ does not contain the profinite completion of any surface group, nor that of any free abelian group of rank greater than 1.*

Remark 7.16 Note that $\widehat{L}$ does contain surface subgroups S of arbitrary large genus (as shown in [12] for example) and free abelian subgroups of arbitrary rank, but the natural map $\widehat{S} \to \widehat{L}$ is never injective. The surface subgroup examples of [12] are in fact dense in $\widehat{L}$.

Next we single out a particular case of an application of the above discussion that connects to two well-known open problems about word hyperbolic groups, namely:

(A) *Does every* 1*-ended word-hyperbolic group contain a surface subgroup?*

(B) *Is every word-hyperbolic group residually finite?*

The first question, due to Gromov, was motivated by the case of hyperbolic 3-manifolds, and in this special case the question was settled recently by Kahn and Markovic [35]. Indeed, given [35], a natural strengthening of (A) above is to ask:

(A′) *Does every* 1*-ended word-hyperbolic group contain a quasi-convex surface subgroup?*

Theorem 7.17 *Suppose that every* 1*-ended word-hyperbolic group is residually finite and contains a quasi-convex surface subgroup. Then there exist no 1-ended word-hyperbolic group* Γ *and free group* F *such that* $\widehat{\Gamma} \cong \widehat{F}$.

Proof Assume the contrary, and let Γ be a counter-example, with $\widehat{\Gamma} \cong \widehat{F}$ for some free group F. Let H be a quasi-convex surface subgroup of Γ. Note that the finite-index subgroups of H are also quasi-convex in Γ. Under the assumption that all 1-ended hyperbolic groups are residually finite, it is proved in [3] that H and all its subgroups of finite index must be separable in Γ. Hence by Lemma 4.6, the natural map $\widehat{H} \to \overline{H} < \widehat{\Gamma}$ is an isomorphism. But as above this yields a contradiction. □

Corollary 7.18 *Suppose that there exists a* 1*-ended word hyperbolic group* Γ *with* $\widehat{\Gamma} \cong \widehat{F}$ *for some free group* F*. Then either there exists a word-hyperbolic group that is not residually finite, or there exists a word-hyperbolic group that does not contain a quasi-convex surface subgroup.*

8 Fuchsian groups, 3-manifold groups and related groups

In this section we prove several results in connection with distinguishing free groups within certain classes of groups. In addition we also prove some results distinguishing 3-manifold groups.

In what follows we denote by $\mathcal{F}$ the collection of Fuchsian groups, and $\mathcal{L}$ the collection of lattices in connected Lie groups.

8.1. In this section we sketch the proof of the following result from [19].

Theorem 8.1 *Let* $\Gamma \in \mathcal{F}$*, then* $\mathcal{G}(\Gamma, \mathcal{L}) = \{\Gamma\}$.

Before commencing with a sketch of the proof, we remark that there exist lattices in connected Lie groups that are not residually finite (recall the discussion at the end of §7.5). For simplicity, in the sketch below we will simply assume all lattices considered are residually finite. This can be bypassed, and we refer the reader to [19] for details on how this is done.

Proof Suppose that $\Delta \in \mathcal{G}(\Gamma, \mathcal{L})$ is residually finite. Then, Corollary 6.7 shows that $b_1^{(2)}(\Delta) \neq 0$. It now follows (see [43] Lemma 1 for example) that Δ fits into a a short exact sequence

$$1 \to N \to \Delta \to F \to 1$$

such that N is finite and F is a lattice in $\mathrm{PSL}(2, \mathbf{R})$.

We next claim that this forces N to be trivial and so Δ is Fuchsian. To see this, suppose that $N \neq 1$. Since N is finite, and Δ is residually finite, it follows that the short exact sequence above can be promoted to a short exact sequence of profinite groups (recall §7.5). Hence the full profinite topology is induced on N by Δ, and we deduce that $\widehat{\Delta}$ contains a non-trivial finite normal subgroup. But, $\widehat{\Delta} \cong \widehat{\Gamma}$, where Γ is a Fuchsian group. This is excluded by the following result proved in [19]. We will not comment on the proof of this result other than to say that it uses profinite group actions on profinite trees. We recall some notation. Write $\mathfrak{cf}(\Gamma)$ to denote the set of conjugacy classes of *maximal finite subgroups* of a group Γ.

Theorem 8.2 *If Γ is a finitely generated Fuchsian group, then the natural inclusion $\Gamma \to \widehat{\Gamma}$ induces a bijection $\mathfrak{cf}(\Gamma) \to \mathfrak{cf}(\widehat{\Gamma})$. More precisely, every finite subgroup of $\widehat{\Gamma}$ is conjugate to a subgroup of Γ, and if two maximal finite subgroup of Γ are conjugate in $\widehat{\Gamma}$ then they are already conjugate in Γ.*

It follows from this that if Γ is a finitely generated non-elementary Fuchsian group, then $\widehat{\Gamma}$ cannot contain a finite non-trivial normal subgroup, since Γ does not.

Given this discussion, to prove Theorem 8.1, it suffices to prove:

Claim: $\mathcal{G}(\Gamma, \mathcal{F}) = \{\Gamma\}$.

Proof of Claim: Suppose that $\Delta \in \mathcal{G}(\Gamma, \mathcal{F})$. If Γ is torsion-free then Δ is torsion-free by Corollary 7.6. Still assuming that Γ is torsion-free, if Γ is a cocompact surface group of genus g then so is Δ. That is to say, Δ cannot be free—this was ruled out by the discussion in §4.7 or Corollary 7.15. In addition, it also cannot be the case that Γ is cocompact and Δ is not (or vice versa). For if this were so, then we could pass to torsion-free subgroups of common finite index that would still have isomorphic profinite completions and this is ruled out by the previous sentence.

If neither Γ_1 nor Γ_2 is cocompact, then each is a free product of cyclic groups. We know that $b_1(\Gamma) = b_1(\Delta)$, and so by Proposition 3.2 the number of infinite cyclic factors in each product is the same. By Theorem 8.2, the finite cyclic factors, being in bijection with the conjugacy classes of maximal finite subgroups, are also the same. Hence the claim follows in this case too.

It only remains to consider the case where both Γ and Δ are cocompact groups with torsion. The genus of Γ is determined by $b_1(\Gamma)$, and so, by Proposition 3.2, Γ and Δ are of the same genus. The periods of Γ and Δ are the orders of representatives of the conjugacy classes of maximal finite subgroups of Γ_i, and so by Theorem 8.2 these must also be the same for Γ and Δ. Thus Γ and Δ have the same signature, and are therefore isomorphic.

This completes the proof of the claim and also Theorem 8.1. □

8.2. In this subsection we focus on proving results distinguishing 3-manifold groups. We summarize this in the following theorem.

Theorem 8.3 *1. Let M be a prime 3-manifold. Then $\pi_1(M)$ is Grothendieck Rigid.*

2. Let Γ be a finitely generated free group of rank $r \geq 2$, and let M be a closed 3-manifold with $\pi_1(M) \in \mathcal{G}(\Gamma)$. Then M is a connect sum of r copies of $S^2 \times S^1$.

3. For $i = 1, 2$, let $M_i = \mathbf{H}^3/\Gamma_i$ where M_1 closed and M_2 non-compact. Then $\widehat{\Gamma}_1$ is not isomorphic to $\widehat{\Gamma}_2$.

4. Let M be a closed hyperbolic 3-manifold and N a geometric 3-manifold. Then if $\pi_1(N) \in \mathcal{G}(\pi_1(M))$, N is a closed hyperbolic 3-manifold.

Proof 1. We have already seen that this holds if M is geometric. Thus we can assume that M is not geometric. Since M is prime, it must therefore have a non-trivial JSJ decomposition. By Theorem 7.3 $\pi_1(M)$ is good. Since M is prime it is aspherical and so we have $H^3(M; \mathbf{F}_p) = H^3(\pi_1(M); \mathbf{F}_p) = \mathbf{F}_p$ for all primes p. On the other hand, if $(\pi_1(M), H)$ is a Grothendieck Pair, where H is finitely generated subgroup of $\pi_1(M)$, then by the discussion in §5.2, H is of infinite index. Moreover, H is also good by Theorem 7.3, and the cover of M corresponding to H, denoted by M_H is still aspherical. However, since this is an infinite sheeted cover, $0 = H^3(M_H; \mathbf{F}_p) = H^3(H; \mathbf{F}_p)$, and hence a contradiction.

2. First, it is clear that $\pi_1(M)$ is infinite. If M is prime, then using the remark at the end of §6, $b_1^{(2)}(\pi_1(M)) = 0$ and the result follows from Corollary 6.4. Thus we can assume that M decomposes as a connect sum $X_1 \# X_2 \# \ldots \# X_s$. Again using the remark in §6 and Example 6.2, we have $s = r$. Also, each X_i has infinite fundamental group since free groups are good and so Lemma 7.6 excludes torsion in the profinite completion.

Now $\pi_1(M)$ has the structure of a free product and so by Lemma 4.8, the profinite topology is efficient. In particular each $\pi_1(X_i)$ is a closed subgroup of $\pi_1(M)$. Suppose that some X_i is not homeomorphic to $S^2 \times S^1$. Then X_i is aspherical, and then either there exists a subgroup $A \cong \mathbf{Z} \oplus \mathbf{Z}$ which is closed in the profinite topology on $\pi_1(X_i)$ and for which the full profinite topology is induced on A (by [57] in the case of Seifert manifolds and [61] for the case where X_i has a non-trivial JSJ decomposition), or there exists a closed surface subgroup of genus > 1 (by [35]) which is closed in the profinite topology and for which the full profinite topology is induced (by [2]). In either case we deduce that $\widehat{\pi_1(M)}$ contains a closed subgroup to which we can apply Corollary 7.13 and deduce a contradiction (by Corollary 7.15).

3. This follows easily from Theorem 7.3 since for all primes p, $H^3(M_2; \mathbf{F}_p) = H^3(\pi_1(M_2); \mathbf{F}_p) = 0$ and $H^3(M_2; \mathbf{F}_p) = H^3(\pi_1(M_2); \mathbf{F}_p) \neq 0$.

4. Since M is closed and hyperbolic, as above, by Theorem 7.3, we can assume that N is closed. It is well known that $\pi_1(M)$ has infinitely many non-abelian finite simple quotients (see [39] for example). Thus we quickly eliminate all possibilities for N apart from those modelled on $\mathbf{H}^2 \times \mathbf{R}$ and $\widetilde{\mathrm{SL}}_2$. In this case, $\pi_1(N)$ has a

description as:

$$1 \to Z \to \pi_1(N) \to F \to 1$$

where Z is infinite cyclic, and F is a cocompact Fuchsian group (we can pass to a subgroup of finite index if necessary so as to arrange the base to be orientable). Since $\pi_1(N)$ is LERF, this short exact sequence can be promoted to (recall the discussion in §4.5):

$$1 \to \widehat{Z} \to \widehat{\pi_1(N)} \to \widehat{F} \to 1.$$

Setting $G = \widehat{\pi_1(N)}$ we have that $G \cong \widehat{\pi_1(M)}$ and so $\pi_1(M)$ is a dense subgroup of G. If $\pi_1(M) \cap \widehat{Z} \neq 1$, then it follows that $\pi_1(M)$ contains an abelian normal subgroup, and this is impossible (as M is a closed hyperbolic 3-manifold). Thus $\pi_1(M) \cap \widehat{Z} = 1$ and therefore $\pi_1(M)$ projects injectively to a dense subgroup of $\widehat{F}$. However, this then contradicts Proposition 6.3. This completes the proof. □

Remarks:

1. Part 1. of Theorem 8.3 was proved in the PhD thesis of W. Cavendish [23] (assuming the then open virtual fibration conjecture for finite volume hyperbolic 3-manifolds).
2. There appears to be no direct proof that distinguishes closed hyperbolic 3-manifolds from finite volume non-compact hyperbolic 3-manifolds by the profinite completions of their fundamental groups. In particular the issue of detecting a peripheral $\mathbf{Z} \oplus \mathbf{Z}$ seems rather delicate.
3. In a similar vein, at present it also seems hard to distinguish a closed prime 3-manifold with a non-trivial JSJ decomposition from a closed hyperbolic 3-manifold by the profinite completions of their fundamental groups. As above the issue of detecting a $\mathbf{Z} \oplus \mathbf{Z}$ is rather subtle.
   However, the author has recently been informed that Wilton and Zalesskii claimed to have now shown that a closed prime 3-manifold with a non-trivial JSJ decomposition from a closed hyperbolic 3-manifold by the profinite completions of their fundamental groups.
4. Funar [26] has shown that there are non-homeomorphic geometric 3-manifolds whose fundamental groups have isomorphic profinite completions. The known examples are torus bundles with SOL geometry. At present, we do not know whether other torus bundles modelled on NIL geometry (which are Seifert fibered), or more generally other Seifert fibered spaces can be distinguished by their finite quotients (even amongst Seifert fibered spaces).

8.3. In this subsection we discuss further properties of a group that is in the same genus as a finitely generated free group. The starting point for this discussion is Section 4 of Peterson and Thom [51] which contains a number of results concerning the structure of finitely presented groups that satisfy their condition ($\star$) and have non-zero $b_1^{(2)}$. We will not state their condition ($\star$) here, but rather remark that the condition is known to hold for left orderable groups and groups that are residually torsion-free nilpotent. We prove the following:

Theorem 8.4 *Let Γ be a finitely presented group in the same genus as a free group F of rank $r \geq 2$. Then:*

(1) *the reduced group C^*-algebra $C^*_\lambda(\Gamma)$ is simple and carries a unique normalised trace.*

(2) *Γ satisfies a the following Freiheitssatz; every generating set $S \subset \Gamma$ has an r-element subset $T \subset S$ such that the subgroup of G generated by T is free of rank r.*

Recall that the *reduced C^*-algebra* $C^*_\lambda(\Gamma)$ is the norm closure of the image of the complex group algebra $\mathbf{C}[\Gamma]$ under the left-regular representation $\lambda_\Gamma : \mathbf{C}[\Gamma] \to \mathcal{L}(\ell^2(\Gamma))$ defined for $\gamma \in \Gamma$ by $(\lambda_\Gamma(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $x \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. A group Γ is C^*-simple if its reduced C^*-algebra is simple as a complex algebra (i.e., has no proper two-sided ideals). This is equivalent to the statement that any unitary representation of Γ which is weakly contained in λ_Γ is weakly equivalent to λ_Γ. We refer the reader to [32] for a thorough account of the groups that were known to be *C^*-simple* by 2006. The subsequent work of Peterson and Thom [51] augments this knowledge.

An important early result in the field is the proof by Powers [55] that non-abelian free-groups are C^*-simple. In contexts where one is able to adapt the Powers argument, one also expects the canonical trace to be the only normalized trace on $C^*_\lambda(\Gamma)$ (cf. Appendix to [16]). By definition, a linear form τ on $C^*_\lambda(\Gamma)$ is a *normalised trace* if $\tau(1) = 1$ and $\tau(U^*U) \geq 0$, $\tau(UV) = \tau(VU)$ for all $U, V \in C^*_\lambda(\Gamma)$. The *canonical trace* is uniquely defined by

$$\tau_{\text{can}}\left(\sum_{f\in F} z_f \lambda_\Gamma(f)\right) = z_e$$

for every *finite* sum $\sum_{f\in F} z_f\lambda_\Gamma(f)$ where $z_f \in \mathbf{C}$ and $F \subset \Gamma$ contains 1.

Proof Note that Γ is necessarily torsion free since $\widehat{\Gamma} \cong \widehat{F}$. By assumption, we have from Corollary 6.7 that $b_1^{(2)}(\Gamma) = r - 1 \neq 0$ and so both parts of the theorem will follow from [51] once we establish that Γ is left orderable (see Corollary 4.6 and 4.7 of [51]). For this we will make use of a result of Burns and Hale [20] that states that if a group Γ is *locally indicable* (i.e., every finitely generated non-trivial subgroup A admits an epimorphism to $\mathbf{Z}$), then Γ is left orderable. Thus the result will follow from the next theorem. Details of the proof will appear elsewhere, we sketch some of the ideas.

Theorem 8.5 *Γ as in Theorem 8.4 is locally indicable.*

Sketch Proof: Let $A < \Gamma$ be a finitely generated non-trivial subgroup. Since Γ is residually finite, A injects in $\widehat{\Gamma} \cong \widehat{F}$ for a finitely generated free group F of rank ≥ 2. Consider the closure $\overline{A} < \widehat{\Gamma}$ which by a slight abuse of notation we view as sitting in $\widehat{F}$. As a closed subgroup we have from Proposition 7.5 that $\text{cd}(\overline{A}) \leq \text{cd}(\widehat{F}) = 1$ (recall Example 7.9). Since $\overline{A} \neq 1$, and $\text{cd}(\widehat{F}) = 1$ we must have that $\text{cd}_p(\overline{A}) = 1$ for some prime p (see Lemma 7.8). The proof is completed by establishing the following claims:

Claim

(1) *There is an epimorphism* $\overline{A} \to \mathbf{Z}/p\mathbf{Z}$.

(2) *The epimorphism* $\overline{A} \to \mathbf{Z}/p\mathbf{Z}$ *in (1) lifts to an epimorphism* $\overline{A} \to \mathbf{Z}_p$.

Given these claims we can now complete the proof that A surjects onto $\mathbf{Z}$. For A being a dense subgroup of $\overline{A}$ must surject all the finite quotients arising from $\overline{A} \to \mathbf{Z}_p \to \mathbf{Z}/p^n\mathbf{Z}$. That is to say A must surject onto $\mathbf{Z}$.

To prove (1) we exploit the fact that $\mathrm{cd}_p(\overline{A}) = 1$ for some prime p, which allows us to conclude that $H^1(\overline{A}; M) \neq 0$ for some finite $\overline{A}$-module M which is p-primary. To prove (2) we use the fact that since $\mathrm{cd}(\overline{A}) = 1$, $\overline{A}$ is a *projective profinite group* (see [56] Chapter 7.6). In particular this allows for lifting problems to be solved, which is needed to pass from (1) to (2). □

Note that fully residually free groups are residually torsion-free nilpotent and non-abelian fully residually free groups have $b_1^{(2)} \neq 0$ (by [17]). As noted above, ($\star$) of [51] applies, and so these groups also satisfy a similar Freiheitssatz.

9 Parafree groups

Recall that a residually nilpotent group with the same nilpotent genus as a free group is called *parafree*. Many examples of such groups are known (see [7], [8] and [10]). Although much is known about finitely generated parafree groups, a good structure theory for these groups is as yet out of reach. Being in the same nilpotent genus as a parafree group, one might wonder about what properties of a free group are shared by a parafree group. For example, in [10], Baumslag asks:

Question 6: *Let G be a finitely generated parafree group and let $N < G$ be a finitely generated, non-trivial, normal subgroup. Must N be of finite index in G?*

This was answered affirmatively in Corollary 6.8 for groups in the same *genus* as a free group, and using similar methods, in [18] we showed this also holds for the nilpotent genus. In particular we showed that if Γ is a finitely generated parafree group in the same nilpotent genus as a free group of rank $r \geq 2$, then $b_1^{(2)}(\Gamma) \geq r - 1$ and in particular is non-zero. Hence the argument given for proving Corollary 6.8 can still be applied. The argument in [18] uses the following variation of Proposition 6.3.

Proposition 9.1 *Let Γ be a finitely generated group and let F be a finitely presented group that is residually-p for some prime p. Suppose that there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ and that $\overline{\Gamma} = \widehat{F}_p$. Then $b_1^{(2)}(\Gamma) \geq b_1^{(2)}(F)$.*

This has various other consequences for parafree groups; for example the reduced group C^*-algebra is simple and carries a unique normalised trace, and recovers Baumslag's result ([8] Theorem 4.1) that parafree groups also satisfy a Freiheitssatz.

We now discuss some other properties of finitely generated parafree groups. In [37], it was shown that a non-abelian finitely presented parafree group is large (i.e., it contains a finite index subgroup that surjects a non-abelian free group). Another property of free groups (which has come to prominence of late through its connections

to 3-manifold topology) is Agol's RFRS condition (see [1]). To define this recall that the rational derived series of a group Γ is defined inductively as follows. If $\Gamma^{(1)} = [\Gamma,\Gamma]$, then $\Gamma_r^{(1)} = \{\, x \in \Gamma : \text{there exists } k \neq 0, \text{ such that } x^k \in \Gamma^{(1)} \,\}$. If $\Gamma_r^{(n)}$ has been defined then define $\Gamma_r^{(n+1)} = (\Gamma_r^{(n)})_r^{(1)}$.

A group Γ is called *residually finite rationally solvable* (RFRS for short) if there is a sequence of subgroups:

$$\Gamma = \Gamma_0 > \Gamma_1 > G_2 \dots$$

such that $\bigcap_i \Gamma_i = 1$, $[\Gamma : \Gamma_i] < \infty$ and $(\Gamma_i)_r^{(1)} < G_{i+1}$.

Theorem 9.2 *Let Γ be a finitely generated parafree group with the same nilpotent genus of a free group of rank $r \geq 2$. Then Γ is RFRS.*

Proof Fix a prime p, and let G denote the pro-p completion of Γ (which by assumption is the free pro-p group of rank $r \geq 2$). Consider the tower of finite index subgroups defined as $P_1(G) = G$, and $P_{i+1}(G) = (P_i(G))^p[G, P_i(G)]$. Note that each $P_i(G)$ is a closed normal subgroup of G, that $\{P_i(G)\}$ forms a basis of open neighbourhoods of the identity, $\bigcap P_i(G) = 1$ and $P_i(G)/P_{i+1}(G)$ is an elementary abelian p-group.

Since $\Gamma \to G$ is injective, we will consider the subgroups $\{\Delta_i = P_i(G) \cap \Gamma\}$. These are then normal subgroups of finite index in Γ that intersect in the identity. RFRS will follow once we show that $(\Delta_i)_r^{(1)} < \Delta_{i+1}$.

To see this, first note that since each quotient Δ_i/Δ_{i+1} is an elementary abelian p-group, then each Δ_i is normal of p-power index in Γ. Hence $\widehat{\Delta}_{i,p} \to \overline{\Delta}_i < G$ is an isomorphism (since Γ is residually p and Δ_i is normal and of p-power index, the full pro-p topology is induced). Hence $\widehat{\Delta}_{i,p}$ is a free pro-p group of rank l say. It follows that Δ_i has first Betti number equal to l (see [18] Corollary 2.9 for example). Moreover, Δ_i and the free group of rank l have the same p-group quotients, and so it follows that $|\mathrm{Tor}(H_1(\Delta_i;\mathbf{Z})|$ is not divisible by p. The proof is completed by the following lemma. □

Before stating and proving this, we make a preliminary remark. If H is a finitely generated group, then trivially $[H,H] < H_r^{(1)}$, and if $H_1(H;\mathbf{Z})$ is torsion-free, then $H_r^{(1)} = [H,H]$. The next lemma is a variation of this.

Lemma 9.3 *Let p be a prime, H be a finitely generated group and K a normal subgroup of H satisfying:*

- *H/K is an elementary abelian p-group.*
- *$|\mathrm{Tor}(H_1(H;\mathbf{Z})|$ is not divisible by p.*

Then $H_r^{(1)} < K$.

Proof As noted above, if $\mathrm{Tor}(H_1(H;\mathbf{Z})) = 1$ then we are done since $H_r^{(1)} = [H,H] < K$. Thus we may suppose that $\mathrm{Tor}(H_1(H;\mathbf{Z})) \neq 1$. Let $x \in H_r^{(1)}$, so that $x^d \in [H,H]$ for some $d \geq 1$. We will assume that $x \notin K$, otherwise we are done. In particular, $d \geq 2$ since $[H,H] < K$ by the first assumption. Hence x projects to a non-trivial element in $H/[H,H]$ and H/K.

Since H/K is an elementary abelian p-group, it follows from the previous discussion that d is divisible by p. On the other hand, the second assumption is that $|\mathrm{Tor}(H_1(H;\mathbf{Z})|$ is not divisible by p. Putting these statements together, it follows that the image of x must have infinite order in $H/[H,H]$, and this is false. In particular we conclude that d cannot be greater than or equal to 2; i.e., $x \in [H,H] < K$ and the lemma is proved. □

Perhaps the most famous open problem about parafree groups is the *Parafreee Conjecture*. This asserts that if Γ is a finitely generated parafree group, then $H_2(G;\mathbf{Z}) = 0$. Although goodness seems like it may be relevant here, it is not quite the right thing—since the nilpotent genus is only concerned with nilpotent quotients. However, a variation is relevant.

One says that a group Γ is *pro-p good* if for each $q \geq 0$, the homomorphism of cohomology groups

$$H^q(\widehat{\Gamma}_p;\mathbf{F}_p) \to H^q(\Gamma;\mathbf{F}_p)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}_p$ is an isomorphism. One says that the group Γ is *cohomologically complete* if Γ is pro-p good for all primes p. Many groups are known to be cohomologically complete. For example finitely generated free groups, RAAG's [42], and the fundamental group of certain link complements in S^3 (see [11]). However, as is pointed out in [18], there are link complements (even hyperbolic) for which the fundamental group is not cohomologically complete. Note that such an example is good by Theorem 7.3.

The connection with the Parafree Conjecture is the following.

Proposition 9.4 *If finitely generated parafree groups are cohomologically complete, then the Parafree Conjecture is true.*

Proof Suppose that Γ is a finitely generated parafree group. Since Γ is parafree, $\widehat{\Gamma}_p$ is a free pro-p group for all primes p. If we now assume that $H_2(\Gamma;\mathbf{Z}) \neq 0$, then for some prime p we must have $H_2(\Gamma;\mathbf{F}_p) \neq 0$. But then the Universal Coefficient Theorem implies that $H^2(\Gamma;\mathbf{F}_p) \neq 0$. If Γ is pro-p good a contradiction is obtained. □

10 Questions and comments

We close with a list of problems and comments motivated by these notes. First, call a finitely generated discrete group *profinitely rigid* if $\mathcal{G}(\Gamma) = \{\Gamma\}$. We begin with various strengthenings of Question 1.

Question 7: *Are finitely generated Fuchsian groups profinitely rigid?*

Question 8: *Are finitely generated Kleinian groups profinitely rigid?*

Restricting to lattices in $\mathrm{PSL}(2,\mathbf{C})$ we can ask by analogy with the hard part of Theorem 8.1:

Question 9: *Let Γ_1 and Γ_2 be lattices in $\mathrm{PSL}(2,\mathbf{C})$. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ is $\Gamma_1 \cong \Gamma_2$?*

Much more ambitious is the following:

Question 10: *Are lattices in rank* 1 *semisimple Lie groups profinitely rigid?*

There is some chance this may be false. In particular, an answer to this question seems closely related to the status of CSP for lattices in $\mathrm{Sp}(n,1)$ ($n \geq 2$). This is also related to the next three questions.

Question 11: *Does there exist a residually finite word hyperbolic group that is not good?*

Question 12: *Does there exist a residually finite torsion free word hyperbolic group* Γ *for which* $\widehat{\Gamma}$ *contains non-trivial elements of finite order?*

Question 13: *Does there exist a residually finite word hyperbolic group that is not highly residually finite?*

Question 14: *Does there exist a word hyperbolic* Γ *for which* $\mathcal{G}(\Gamma)$ *contains another word hyperbolic group?*

Using Proposition 5.1(2) Grothendieck Pairs (Γ, N) can be constructed so that Γ is word hyperbolic. However, in the known examples, N is not word hyperbolic.

As discussed in §3.4, there are lattices of higher rank for which the genus contains more than one element. However, some interesting special cases seem worth considering.

Question 15: *Is* $\mathrm{SL}(n,\mathbf{Z})$ *profinitely rigid for all* $n \geq 3$*? Is* $\mathrm{SL}(n,\mathbf{Z})$ *Grothendieck Rigid for all* $n \geq 3$*?*

Note that using [15] and [54], for large enough n examples of subgroups $H < \Gamma < \mathrm{SL}(n,\mathbf{Z})$ can be constructed so that (Γ, H) is a Grothendieck Pair.

Motivated by the Parafree Conjecture and a desire to have some type of structure theory for finitely generated parafree groups we raise:

Question 16: *Are finitely generated parafree groups cohomologically complete? How about good?*

We saw in Theorem 9.2 that finitely generated parafree groups are RFRS. The RFRS property holds for groups that are *special* (see [1]). That parafree groups are special seems too much to ask, however, the following seems plausible:

Question 17: *Are finitely generated parafree groups virtually special?*

Note that a positive answer to Question 17 would also imply that finitely generated parafree groups are linear. This is still an open question (see [10] Question 8).

On a slightly different topic. Let Γ_g denote the Mapping Class Group of a closed orientable surface of genus $g \geq 2$.

Question 18: *Is Γ_g profinitely rigid?*

Question 19: *Is Γ_g good?*

This question was raised in [38] in connection with the geometry of moduli spaces of curves of genus g. As pointed out in [38], the answer is known for $g \leq 2$ (the case $g = 1$ follows from Theorem 7.3).

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