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ARITHMETIC KNOTS IN CLOSED 3-MANIFOLDS*

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1 Introduction

Let O_d denote the ring of integers in $\mathbf{Q}(\sqrt{-d})$. An orientable finite volume cusped hyperbolic 3-manifold M is called **arithmetic** if the faithful discrete representation of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbf{C})$ is conjugate to a group commensurable with some Bianchi group $\mathrm{PSL}(2,O_d)$. If M is a closed orientable 3-manifold, we say a link $L \subset M$ is arithmetic if $M \setminus L$ is arithmetic. Since the figure-eight knot complement is well-known to be arithmetic [19] and universal [14], it follows that every closed orientable 3-manifold contains an arithmetic link. In the case of S^3, the figure-eight knot is the only arithmetic knot [18]. A natural question therefore is; does every closed orientable 3-manifold contain an arithmetic knot? One motivation for this question is that if every closed orientable 3-manifold contained an arithmetic knot, this would imply the Poincare Conjecture. For the methods of [18] show that the figure-eight knot in S^3 is the only arithmetic knot in a homotopy 3-sphere. Our main result is the following:

Theorem 1.1 *There exist closed orientable 3-manifolds which do not contain an arithmetic knot.*

Our methods give much more precise versions of Theorem 1.1, particularly for non-hyperbolic 3-manifolds. For example, for certain Lens Spaces, we can give fairly complete statements. By Lens Space we shall always mean a closed orientable 3-manifold of Heegaard genus 1 and finite fundamental group. Thus we exclude $S^2 \times S^1$. There are well-known examples of arithmetic knots in Lens Spaces; namely the double cover of the figure eight knot complement (a knot in $L(5,2)$) and the sister of the figure eight knot complement (a knot in $L(5,1)$). We show (see §2 for definitions):

Theorem 1.2 *Let L be a Lens Space, and assume that L contains a knot K derived from a quaternion algebra then $L \setminus K$ is homeomorphic to the sister of the figure eight knot complement or the double cover of the figure eight knot complement.*

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The figure-eight knot complement admits no Lens Space filling ([21]), however the sister of the figure eight knot complement has three such fillings, $L(5,1)$ (twice), and $L(10,3)$ (see [6] for example). Using standard properties about invariant trace fields (see §2 for more details) we have the following corollary of Theorem 1.2 providing many examples for Theorem 1.1.

Corollary 1.3 *Let L be a Lens Space, with $|\pi_1(L)| = r$ of odd order. If $r \neq 5$, then L does not contain an arithmetic knot.*

The assumption on odd order is necessary, since $\mathbf{RP}^3$ *does* contain an arithmetic knot (see Table 1). This occurs as -2-surgery on one component of the Whitehead link (see [5]) and is not derived from a quaternion algebra.

Little seems known about the set of 1-cusped arithmetic hyperbolic 3-manifolds. We suspect that they are very rare. However, at present it is unknown whether there are only finitely many commensurability classes of 1-cusped arithmetic hyperbolic 3-manifolds. Some discussion of what is known is contained in §5 and 6.

The case of links of at least 2 components is very different. In this generality there are infinitely many arithmetic links in S^3. Indeed the first author recently observed [3], that every knot can be realized as a component of an arithmetic link in S^3.

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2 Preliminaries

Here we collect some well-known facts.

2.1

Recall that if Γ is a Kleinian group, then the invariant trace-field of Γ, denoted by $k\Gamma$, is the field $\mathbf{Q}((\mathrm{tr}\gamma)^2 \ : \ \gamma \in \Gamma)$, see [17]. An equivalent reformulation of arithmeticity in the setting of cusped manifolds is (see [18] for example):

Theorem 2.1 *Let $M = \mathbf{H}^3/\Gamma$ be a cusped hyperbolic 3-manifold of finite volume. Then M is arithmetic if and only if $k\Gamma = \mathbf{Q}(\sqrt{-d})$ for some d, and $\mathrm{tr}^2\gamma \in O_d$ for all $\gamma \in \Gamma$.* □

When M is a link complement in a $\mathbf{Z}_2$-homlogy 3-sphere, it follows from Corollary 2.3 of [17] that the trace-field coincides with the invariant trace-field, so we have,

Corollary 2.2 *Let $M = \mathbf{H}^3/\Gamma$ be a link complement in a $\mathbf{Z}_2$-homology 3-sphere. Then M is arithmetic if and only if there is a d such that $\mathrm{tr}\gamma \in O_d$ for all $\gamma \in \Gamma$.* □

In the notation of [18] and [17], when the conclusion of Corollary 2.2 holds, then Γ (or M) is *derived from a quaternion algebra.* This is equivalent to Γ being conjugate into $\mathrm{P}\mathcal{O}^1$ for an order $\mathcal{O} \subset M(2, \mathbf{Q}(\sqrt{-d}))$.

As an extension to the notion of an arithmetic knot, we say a knot $K \subset M$ is derived from a quaternion algebra if $M \setminus K = \mathbf{H}^3/\Gamma$, and Γ is derived from a quaternion algebra.

Note that Corollary 1.3 is now immediate from Theorem 1.2 and Corollary 2.2.

2.2

We now fix some notation that will be maintained throughout, and make some standard deductions about arithmetic knots in 3-manifolds. In what follows, $Q_d = \mathbf{H}^3/\mathrm{PSL}(2, O_d)$.

Proposition 2.3 *Let M be a closed orientable 3-manifold and $K \subset M$ a knot derived from a quaternion algebra. Then, $M \setminus K \to Q_d$ where $\mathbf{Q}(\sqrt{-d})$ has class number 1. Furthermore, if M is a rational homology 3-sphere, then*

$$d \in \{1, 2, 3, 7, 11, 19\}.$$

Proof: Let $M \setminus K = \mathbf{H}^3/\Gamma$. By assumption and from the remarks in §2.1, there is a maximal order $\mathcal{O}$ of $M(2, \mathbf{Q}(\sqrt{-d}))$, such that $\Gamma < P\mathcal{O}^1$. By Lemma 2 of [18], if the class number of $\mathbf{Q}(\sqrt{-d})$ is greater than 1, any group $P\mathcal{O}^1$ has more then one cusp. By assumption $M \setminus K$ has only 1 cusp, so we are forced to have the class number of $\mathbf{Q}(\sqrt{-d})$ equal 1. In this case there is only one type of maximal order, and so we can assume without loss of generality that $\Gamma < \mathrm{PSL}(2, O_d)$ of finite index. This proves the first part.

If M is a rational homology 3-sphere, then $M \setminus K$ has trivial cuspidal cohomology. In this case the solution to the Cuspidal Cohomology Problem ([22]) allows us to reduce to those d given by $d \in \{1, 2, 3, 7, 11, 19\}$. □

An immediate corollary of Proposition 2.3 and Lemma 2.2 is

Corollary 2.4 *If M is a $\mathbf{Z}_2$-homology 3-sphere and $K \subset M$ arithmetic, then $M \setminus K \to Q_d$ and $d \in \{1, 2, 3, 7, 11, 19\}$.* □

We make some additional comments. Let M be a closed orientable 3-manifold and assume we have a finite cover

$$M \setminus K = \mathbf{H}^3/\Gamma \to Q_d.$$

Then since $\mathrm{PSL}(2, O_d)$ obviously contains parabolic elements fixing ∞, there is a parabolic element μ in Γ fixing ∞ which is a "meridian" of K, in the sense that trivially filling $M \setminus K$ along μ gives back M. Let $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, for some $x \in O_d$. This notation will be fixed throughout.

2.3

We will make use of the following elementary fact about horoballs.

Lemma 2.5 *Suppose Γ is a non-elementary Kleinian group containing a parabolic element fixing ∞. If for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$ we have $|c| \geq 1$, then a maximal horoball at infinity has Euclidean height at most 1.*

Proof.
Let $\mathcal{H}$ be a maximal horoball at infinity at Euclidean height h say. An elementary calculation shows that the image of $\mathcal{H}$ under an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ is a horoball based at a/c of diameter $\frac{1}{h|c|^2}$. Thus if this ball is also maximal, then $h = \frac{1}{h|c|^2}$ and so $h = 1/|c|$. Since $|c| \geq 1$ the lemma follows. □

As an application of this lemma we have the following which will be useful to us (this has also been observed by I. Agol).

Theorem 2.6 *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold with a single cusp. Assume Γ contains the elements $\alpha = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ where:*

1. *α and β are conjugate in Γ, and*
2. *$|u| = |v| = 1$.*

Then M is homeomorphic to the figure-eight knot complement in S^3.

Proof. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$. Then Jørgenson's Inequality applied to T and α gives,

$$|\mathrm{tr}[\alpha, T] - 2| = |uc|^2 \geq 1.$$

By assumption 2, we deduce that $|c| \geq 1$. Thus Lemma 2.5 implies the height of a maximal horoball at infinity is at most 1.

An element conjugating α to β has the form $T = \begin{pmatrix} 0 & i\sqrt{\frac{u}{v}} \\ i\sqrt{\frac{v}{u}} & \delta \end{pmatrix}$ for some $\delta \in \mathbf{C}$. As noted in the proof of Lemma 2.5, if the maximal horosphere is at height h then the image of this horosphere under T, S say, has height $\frac{1}{h|v/u|}$. Since $|u| = |v| = 1$, S has height $1/h$. The height of a maximal cusp is at most 1, with $h \leq 1$. Putting these statements together, we see that $h = 1$ in this case, with S being maximal.

Recall that the waist size of a 1-cusped hyperbolic 3-manifold is just the length of the shortest translation in a maximal cusp. From above the maximal cusp is at height 1, so we deduce the length of α is 1. Since the waist size is at least one ([1]), the waist size of the manifold M must be exactly 1. However, a result of Adams [1], says that the unique hyperbolic 3-manifold with a maximal cusp of waist size 1 is the complement of the figure-eight knot. □

3 Lens Spaces

In this section we establish Theorem 1.2.

3.1 Proof of Theorem 1.2:

Assume that K is a knot in L which is derived from a quaternion algebra. Let $L \setminus K = \mathbf{H}^3/\Gamma$. By Proposition 2.3, and the remarks after Corollary 2.4, we have $\Gamma < \mathrm{PSL}(2, O_d)$, with $d \in \{1, 2, 3, 7, 11, 19\}$, and a parabolic element $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in Γ fixing ∞ which is a meridian of K.

The argument now breaks up into two cases, depending on whether x is, or is not, a unit in O_d.

Case 1: x is a unit

We begin with a preliminary comment. Since Γ has finite index in $\mathrm{PSL}(2, O_d)$, there is a parabolic element $\nu \in \Gamma$ which is a Γ-conjugate of μ fixing 0. Such an element has the form $\nu = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. Now y is also a unit. To see this, note ν is Γ-conjugate to μ and so there is an element of $\mathrm{PSL}(2, O_d)$ whose form is that of the element T given in the proof of Theorem 2.6. The off-diagonal entries must therefore be units, and direct calculation shows y is a unit.

Using the fact that the only quadratic imaginary units are ± 1, $\pm i$, $\pm\omega$ and $\pm\overline{\omega}$ where $\omega^2 + \omega + 1 = 0$, μ is one of the following elements:

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm\omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm\overline{\omega} \\ 0 & 1 \end{pmatrix}$$

and ν is one of the following elements,

$$\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm\omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm\overline{\omega} & 1 \end{pmatrix}.$$

In particular we are in the situation of Theorem 2.6, and so we deduce $L \setminus K$ is homeomorphic to the complement of the figure eight-knot in S^3. But there is no Lens Space filling on the figure-eight knot complement ([21]), so this completes the proof. □

Remark: In the case of $d \neq 1, 3$, an elementary argument can be used. In these cases, there are no units apart from ± 1, so that it is easy to see from above that Γ contains the group

$$< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} >$$

which is simply $\mathrm{PSL}(2, \mathbf{Z})$. But Γ is torsion free, so this is a contradiction.

Case 2: x is not a unit

Since $x \in O_d$ is not a unit $I =< x >$ is a non-trivial ideal of O_d. For an ideal $B \subset O_d$, let $\Gamma(B)$ denote the principal congruence subgroup of $\mathrm{PSL}(2, O_d)$ of level B.

Associated to the S^3 covering of L we have a link $J \subset S^3$ and a cyclic cover $S^3 \setminus J \to L \setminus K$. Notice that J has at least 2 components, since if not, this would force J to be the figure-eight knot and it is well-known that this knot complement cannot cover (non-trivially) any other hyperbolic 3-manifold.

Let $S^3 \setminus J = \mathbf{H}^3/\Gamma_J$. Note that $\mu \in \Gamma(I)$ by hypothesis, and $\mu \in \Gamma_J$ by definition of the covering. Furthermore since Γ_J is generated by Γ conjugates of μ, and these will also lie in $\Gamma(I)$ we deduce that $\Gamma_J < \Gamma(I)$. Thus we have the diagram of coverings shown below:

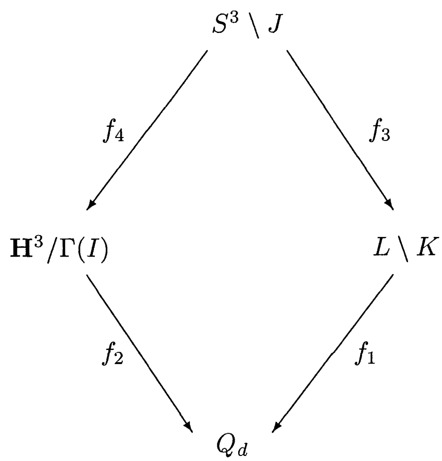

Figure 1

Consider the cyclic covering $f_3 : S^3 \setminus J \to L \setminus K$. Let $M \in \Gamma$ be an element generating this cyclic covering group. We have:

$$\Gamma \;=\; <\Gamma_J, M> \;\subset\; <\Gamma(I), M>.$$

Since $L \setminus K$ has one cusp, if we let $\Delta(I) =< \Gamma(I), M >$, then $\mathbf{H}^3/\Delta(I)$ has one cusp. As we now show, this situation is very rare.

Lemma 3.1 *In the notation above, $\mathbf{H}^3/\Delta(I)$ does not have one cusp except possibly in the following cases:*

1. *$d \in \{1, 2, 7\}$, I is a prime ideal of norm 2 and $\mathbf{H}^3/\Gamma(I)$ has 3 cusps.*
2. *$d = 3$, I is a prime ideal of norm 4, and $\mathbf{H}^3/\Gamma(I)$ has 5 cusps.*
3. *$d = 1$, $I =< 1 \pm i >^n$, where $n = 2, 3$ and $\mathbf{H}^3/\Gamma(I)$ has 6 or 12 cusps respectively.*
4. *$d = 7$, $I =< \frac{(1\pm\sqrt{-7})}{2} >^2$, and $\mathbf{H}^3/\Gamma(I)$ has 6 cusps.*

Deferring the proof of Lemma 3.1 until §3.2, we complete the proof of Theorem 1.2.

We begin with an observation (in this setting) which is implicit in the proof of Lemma 4 of [18].

If G is a group, we let $< b >_G$ denote the normal closure of the element $b \in G$ in G. We claim:

Claim: In the setting of cases 1, 3 and 4 of Lemma 3.1

$$< \mu >_\Gamma = < \mu >_{\mathrm{PSL}(2,O_d)} .$$

To establish this claim, we fix some notation. Let P denote the peripheral subgroup of $\mathrm{PSL}(2, O_d)$ fixing ∞. Since $L \setminus K$ and Q_d both have 1-cusp it follows as in [18] that $\mathrm{PSL}(2, O_d) = P.\Gamma$. Note if $d \neq 1, 3$, the only elements fixing ∞ are translations and so μ commutes with these. In the case of $d = 1$, the element $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ fixes ∞ and conjugates μ to μ^{-1}. With these statements we see:

$$< \mu >_{\mathrm{PSL}(2,O_d)} = < \mu >_{P.\Gamma} = < \mu >_\Gamma,$$

as required.

Remark: Note this argument does not a priori work in the case of $d = 3$ since it is not clear that

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \mu \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$$

lies in $< \mu >_\Gamma$.

Returning to the proof of Theorem 1.2, since $< \mu >_\Gamma = \Gamma_J$, we have, in particular $< \mu >_{\mathrm{PSL}(2,O_d)}$ has finite index in $\mathrm{PSL}(2, O_d)$.

d= 2: In this case since there are no non-trivial units, we can assume $x = \sqrt{-2}$. A presentation for $\mathrm{PSL}(2, O_2)$ is given in [20]

$$< a, b, \mu \mid b^2 = (ab)^3 = (b^{-1}\mu^{-1}b\mu)^2 = 1, [a, \mu] = 1 > .$$

Setting $\mu = 1$, we obtain $\mathbf{Z}_2 * \mathbf{Z}_3$ which is infinite and so a contradiction.

d = 1, 7: We will make use of Magma ([8]) in our considerations. Presentations for $\mathrm{PSL}(2, O_1)$ and $\mathrm{PSL}(2, O_7)$ can be found in [13] or [20]:

$$\mathrm{PSL}(2, O_1) = < a, \ell, t, u \mid [t, u] = \ell^2 = (t\ell)^2 = (u\ell)^2 = (a\ell)^2 = a^2 = (ta)^3 = (ua\ell)^3 = 1 >,$$

$$\mathrm{PSL}(2, O_7) = < a, b, c \mid b^2 = (ab)^3 = (bac^{-1}bc)^2 = 1, [a, c] = 1 > .$$

Consider first case (1) in Lemma 3.1 with $d = 1$ or 7. That is $I = < 1 + i >$ or $< \frac{1 \pm \sqrt{-7}}{2} >$. In these cases the index $[\mathrm{PSL}(2, O_d) : < \mu >_\Gamma]$ is finite and in fact we get as quotient groups S_4 and S_3 respectively (as can be checked directly from the presentation using Magma). The minimal index of a torsion-free subgroup in these

Bianchi groups is 12 and 6 respectively, see [13]. Since f_3 is a covering of degree > 1, this rules out the case $d = 7$ immediately. For the case $d = 1$, we deduce from the above remarks that $[\mathrm{PSL}(2, O_1) : \Gamma] = 12$. However, the minimal index torsion free subgroups in $\mathrm{PSL}(2, O_1)$ all have 2 cusps, see [7] or [13].

We now consider the last two cases. First assume that $I = < 2 >$ or $< 2 \pm 2i >$. So that up to sign $x = 2, 2i$ or $(2 \pm 2i)$. To complete the proof in this case, we shall show that for these values of x, $\mathrm{PSL}(2, O_1)/ < \mu >_{\mathrm{PSL}(2,O_1)}$ is infinite and this will be the desired contradiction. Again we make use of Magma. We wish to set $\mu = 1$ for μ as above. In terms of the given presentation this means setting one of $t^2 = 1$, $u^2 = 1$, $t^2u^2 = 1$ and $t^2u^{-2} = 1$. We can then check directly using Magma that the groups $\mathrm{PSL}(2, O_1)/ < \mu >_{\mathrm{PSL}(2,O_1)}$ all have subgroups of index 6 with infinite abelianization, hence the required quotient groups are infinite.

When $d = 7$, we can argue as follows. With the ideal $I = < \frac{(1 \pm \sqrt{-7})}{2} >^2$, the only possibilities for x (up to sign) are $x = \frac{(-3 \pm \sqrt{-7})}{2}$, and so

$$\mu = \begin{pmatrix} 1 & \frac{(-3 \pm \sqrt{-7})}{2} \\ 0 & 1 \end{pmatrix}.$$

These can be written as $a^{-2}c$ and $a^{-1}c^{-1}$ in the above generators. Setting these equal to 1 in the above presentation produces a group of order 48 (again using Magma for example). The image of $< a, c >$ in this quotient group has order 8. Thus we deduce that in either of these cases $S^3 \setminus J = \mathbf{H}^3/ < \mu >_{\mathrm{PSL}(2,O_d)}$ has 6 cusps, and the cyclic cover f_3 has degree 6. However, this forces the covering f_1 in this case to be degree 8. However, $\mathrm{PSL}(2, O_7)$ has no torsion-free subgroup of index 8 (see [13]).

We are therefore reduced to the second case of Lemma 3.1. Let $\Delta = \Delta(I)$, then for $\mathbf{H}^3/\Delta$ to have one cusp, the only possibility is that the cover $\mathbf{H}^3/\Gamma(2) \to \mathbf{H}^3/\Delta$ is a regular cyclic cover of degree 5, with the cyclic group permuting the cusps. Furthermore $\mathbf{H}^3/\Gamma(2)$ is a manifold (it is a link complement in S^3), and so Δ cannot contain elements of finite order since the only elliptic elements in $\mathrm{PSL}(2, O_d)$ are of orders 2 and 3. Thus Δ is torsion-free and index 12 in $\mathrm{PSL}(2, O_3)$. There are only two such conjugacy classes of groups, [13]; being represented by the fundamental groups of the figure-eight knot complement and its sister. However, the fundamental group of the figure-eight knot complement does not contain $\Gamma(2)$. To see this, suppose that Δ is the fundamental group of the figure-eight knot complement. Then $\Gamma(2)$ is a normal subgroup of Δ of index 5. For homological reasons, the figure-eight knot has a unique such covering, and this has only one cusp. This is a contradiction. Thus $\mathbf{H}^3/\Delta = X$ is homeomorphic to the sister of the figure eight-knot.

We now consider the covering $g : L \setminus K \to X$. We claim the following:

Claim: g extends to a cover $g' : L \to X(s)$ where for some Dehn filling $X(s)$ of X.

Deferring the proof of this claim for the moment we complete the proof of Theorem 1.2. Since L is finite cyclic, $X(s)$ has finite fundamental group. There are 5 such fillings on the sister of the figure eight knot (see [6] for example). These are $L(5, 1)$

(twice), $L(10,3)$, and two fillings giving a manifold with fundamental group $T \times \mathbf{Z}_5$, where T is the binary tetrahedral group.

The case of $L(5,1)$ just gives manifolds homeomorphic to the sister (the covering degree of $g' = 1$). $L(10,3)$ gives examples where the covering degree is 2 and the manifold is homeomorphic to the unique double cover of the figure eight knot and its sister.

To handle the final case, we again make use of Magma. In the notation above X is the sister of the figure eight knot, and using its description as a once-punctured torus bundle over the circle, a presentation for $\pi_1(X)$ is:

$$< a, b, t \mid t^{-1}at = aba^{-1}, t^{-1}bt = b^{-3}a^{-1} >,$$

with a framing for $\pi_1(\partial X)$ being $< t, \ell >$ where $\ell = aba^{-1}b^{-1}$. In this framing the two fillings giving manifolds with fundamental group $T \times \mathbf{Z}_5$, are $(3,1)$ and $(3,2)$.

We now fix attention on $(3,1)$-filling, denoted M below, the argument in the other case is exactly the same. Let $\phi : \pi_1(X) \to \pi_1(M) = T \times \mathbf{Z}_5$ be the surjective homomorphism induced by $(3,1)$-filling. A calculation with Magma (for example) shows that $\phi(\pi_1(\partial X))$ is a cyclic group C of order 6. Hence the coverings of X determined by C and ker ϕ both have 20 cusps. Now we are assuming (by the claim) that there is a covering $L \to M$, with L a Lens Space, and knots K and k with $L \setminus K \to M \setminus k \cong X$. Notice also that the covering Y of X determined by ker ϕ covers $L \setminus K$. Since Y has 20 cusps, the only possiblity for $\pi_1(L)$ is a cyclic group of order 20, 40 or 60. Now it is easy to see that the group $T \times \mathbf{Z}_5$ has no subgroup of index 2, and has a unique subgroup of index 3 which is not cyclic. Thus it suffices to check whether X has any 6-fold covers which are knots in Lens Spaces. One can check using Snap Pea ([23]) for instance that this is not the case. There is a 2-cusped example arising from a Lens Space L with $\pi_1(L) = 20$. □

Proof of Claim: We have $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \Gamma$ and by Lemma 3.1 $|x| = 2$. Thus $\mu = \begin{pmatrix} 1 & 2u \\ 0 & 1 \end{pmatrix}$ where $u \in \{\pm 1, \pm\omega, \pm\omega^2\}$. Such an element lies in $\Gamma(2)$, and therefore $\Delta(2)$. Note this element is primitive in $\Delta(2)$. For if not, then the only possibilities are $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with u as above. However $\Delta(2)/\Gamma(2)$ is cyclic of order 5 so this implies $\begin{pmatrix} 1 & 5u \\ 0 & 1 \end{pmatrix} \in \Gamma(2)$ which is clearly absurd. Now filling X and $L \setminus K$ along curves in their boundaries determined by $\mu = \begin{pmatrix} 1 & 2u \\ 0 & 1 \end{pmatrix}$ extends the cover. □

3.2

We now prove Lemma 3.1 by showing that for those $d \in \{1, 2, 3, 7, 11, 19\}$, it is impossible to get a 1-cusped orbifold as a quotient of $\mathbf{H}^3$ by adjoining a single element to $\Gamma(I)$, apart possibly from the cases listed in Lemma 3.1. We recall some facts that we will appeal to:

1. The ramification theory of primes in quadratic imaginary number fields implies that $|\mathbf{F}| = \ell$ or ℓ^2 where ℓ is the characteristic of $\mathbf{F}$ (see [16]). Let

$$\theta_{\mathcal{P}} \ : \ \mathrm{PSL}(2, O_d) \to \mathrm{PSL}(2, O_d/\mathcal{P})$$

denote the induced homomorphism obtained reduction modulo $\mathcal{P}$.

2. If $\mathbf{F}$ is a finite field of $q = \ell^n$ elements then (see [10]):

$$|\mathrm{PSL}(2,\mathbf{F})| = \begin{cases} \frac{(q^2-1)q}{2} & ,\ell \neq 2 \\ (q^2-1)q & ,\ell = 2. \end{cases}$$

3. Unipotent elements of $\mathrm{PSL}(2,\mathbf{F})$ (with $\mathbf{F}$ as in 2) have order ℓ, and non-unipotent elements in $\mathrm{PSL}(2,\mathbf{F})$ (with $\mathbf{F}$ as in 2) have orders at most $\frac{q+1}{2}$ or $q+1$ dependent on whether $\ell \neq 2$ or $\ell = 2$ ([10]).

Consider first the case when $I = \mathcal{P}$ a prime ideal of O_d of residue field of characteristic ℓ. When $d \neq 1,3$ a peripheral subgroup P_d of $\mathrm{PSL}(2, O_d)$ fixing ∞ consists only of translations and so is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$. Thus $\theta_{\mathcal{P}}(P_d)$ is cyclic of order ℓ or abelian non-cyclic of order ℓ^2. In which case, we see that the number of cusps of $\Gamma(\mathcal{P})$ is $q^2 - 1$ or $\frac{q^2-1}{2}$ (dependent on whether $\ell = 2$ or not) where $q = \ell$ or ℓ^2.

We are assuming that a single element M is adjoined to $\Gamma(\mathcal{P})$ resulting in a group with one conjugacy class of peripheral subgroups. Thus M acts cyclically on the cusps of $\mathbf{H}^3/\Gamma(\mathcal{P})$ identifying them to one. However, this is impossible when $\mathcal{P}$ is a prime of norm at least 3 (in the cases $d \neq 1,3$). To see this, consider the case where $\ell \neq 2$, then from above $\Gamma(\mathcal{P})$ has $\frac{q^2-1}{2}$ cusps which are identifed by M. But M has order q or order at most $\frac{q+1}{2}$. It is then easy to check the conclusion on the norm of $\mathcal{P}$ holds. A similar argument applies when $\ell = 2$. That $d = 2,7$ of (1) of Lemma 3.1 are the only allowable $d's$ (other than $d = 1,3$) is completed by observing that there is no prime of norm 2 in O_{11} or O_{19}. The number of cusps given can be computed from above.

In the case when $d = 1,3$, the group P_d of $\mathrm{PSL}(2, O_d)$ fixing ∞ has additional elliptic elements. This adds some complication to the above analysis but arguing as above gives in these cases bounds on the norm of $\mathcal{P}$ of at most 5 ($d = 1$) and 7 ($d = 3$). In O_1 there are no primes of norm 3, but a prime of norm 2 generated by $(1+i)$ and two of norm 5, $(2 \pm i)$. For $d = 3$, there are no primes of norm 2, or 5 but a prime of norm 3 generated by $\sqrt{-3}$ and two of norm 7, generated by $(2 \pm \sqrt{3})$. Now from above $\Gamma(< 2 \pm i >)$ has 6 cusps, and the quotient group $\mathrm{PSL}(2, O_1)/\Gamma(< 2 \pm i >) \cong A_5$. But there is no element of order 6 in A_5 and so there cannot be an element M as claimed. A similar argument works for $d = 3$ in the case of a prime ideal of norm 3 (there is no element of order 4 in A_4) and norm 7 (there is no element of order 8 in $\mathrm{PSL}(2, \mathbf{F}_7)$). This completes (1) and (2) of Lemma 3.1.

Now suppose that I is not prime. If $< M, \Gamma(I) >$ has one cusp, then for any prime $\mathcal{P}$ with $\mathcal{P}|I$ it follows that $< M, \Gamma(\mathcal{P}) >$ has one cusp. Thus we are reduced to considering products of powers of the prime ideals given in (1) and (2) of Lemma 3.1. Consider the case of $d = 1$, in which case we must deal with powers of the ideal $\mathcal{P} =< 1+i >$. Now $\mathrm{PSL}(2, O_1)/\Gamma(\mathcal{P}) \cong S_3$ and for each positive integer t, $\Gamma(\mathcal{P}^t)/\Gamma(\mathcal{P}^{t+1}) \cong \mathbf{Z}_2^3$. Using these facts and a description of $\theta_{\mathcal{P}}(P_1)$ one may check that $\Gamma(\mathcal{P})$ has 3 cusps, $\Gamma(\mathcal{P}^2)$ has 6 cusps, $\Gamma(\mathcal{P}^3)$ has 12 cusps and $\Gamma(\mathcal{P}^4)$ has 96 cusps. However, using the structure of the quotient groups discussed above, there cannot be an element of order 96 in $\mathrm{PSL}(2, O_1)/\Gamma(\mathcal{P}^4)$ which would induce an isometry cyclically permuting all the cusps. Note this naive argument fails for the exponent of $\mathcal{P}$ being 2 and 3. Hence all exponents ≥ 4 can be eliminated.

The other cases of Lemma 3.1 are handled similarly. □

Remark: The arguments above actually show more. For instance the hypothesis that L is a Lens Space can be weakened to L simply having finite cyclic fundamental group. Going through the proof it can be seen that S^3 can be replaced by a homotopy 3-sphere, since all that is used are the facts that a link group in a homotopy 3-sphere is normally generated by a meridian, and that an arithmetic knot complement in a homotopy 3-sphere is the figure eight knot complement in S^3 (by [18]).

4 Other non-hyperbolike constructions

Although not as complete as the results for Lens Spaces, we have results for some other classes of manifolds.

4.1

An important role in some of what follows is played by the 2π-Theorem of Gromov and Thurston (see [12] and [4]). In fact it will be convenient to use the more refined version due to Agol and Lackenby, [2],[15]. Recall the terminology of [11], that a closed orientable 3-manifold is called *hyperbolike* if it is irreducible, and has infinite fundamental group containing no $\mathbf{Z} \oplus \mathbf{Z}$.

Theorem 4.1 *Let M be a finite volume hyperbolic 3-manifold with a single cusp. Let C be a horospherical cusp torus. If α is a slope on C whose length (as measured on C) is at least* 6, *then the manifold obtained by α-Dehn filling on M is hyperbolike.* □

4.2

Throughout this section we let M be a closed orientable non-hyperbolike 3-manifold, and $K \subset M$ a knot derived from a quaternion algebra. As before we get a group $\Gamma < \mathrm{PSL}(2, O_d)$ (with d as before) of finite index with $\mathbf{H}^3/\Gamma = M \setminus K$ and $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ a meridian.

Lemma 4.2 $|x| < 6$.

Proof.

Since M is not hyperbolike, so Theorem 4.1 implies the length of the meridian on any horospherical cusp cross-section is less than 6. Consider a maximal cusp of

$L \setminus K$, and lift it to $\mathbf{H}^3$. A point of tangency of the maximal horoball at infinity, denoted by $\mathcal{H}$, occurs at some height h. Note that since $\Gamma < \mathrm{PSL}(2, O_d)$ any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ not commuting with μ satisfies $|c| \geq 1$. Hence Lemma 2.5 implies $h \leq 1$.

By definition of the hyperbolic metric, the length of μ measured on the horosphere $\mathcal{H}$ is $|x|/h$. Since $h \leq 1$, and we require this length to be less than 6, it follows that $|x| < 6$. □

Remark: Note that there are only a finite number of quadratic imaginary integers satisfying the bound in Lemma 4.2. Indeed, the values of d given in §2.2 restrict this even more.

Using Lemma 4.2 we can prove a variety of non-existence results for arithmetic knots in non-hyperbolike 3-manifolds. For example,

Theorem 4.3 *Let M be a closed orientable 3-manifold whose fundamental group is isomorphic to that of a connect sum of Lens Spaces L_i of prime order p_i with $p_i > 31$ for $p_1, \ldots, p_m$.*

Then M does not contain an arithmetic knot.

Proof

Note that by assumption M is a $\mathbf{Z}_2$ homology 3-sphere. Hence Corollary 2.4 reduces us to the case when the knot is derived from a quaternion algebra and we are in the setting of §2.2.

The case when x is a unit is handled by Theorem 2.6, with the extra observation that we deduce that M could be one of the non-hyperbolic fillings on the figure-eight knot. However, in this case there are no reducible surgeries, [21].

Thus, assume $x \in O_d$ is not a unit, and so as before, there is a prime ideal $\mathcal{P}$ of O_d with $\mathcal{P} | < x >$. Consider the homomorphism $\theta_{\mathcal{P}}$ from §3.2 restricted to Γ. By construction $\theta_{\mathcal{P}}(\mu) = 1$, and so we induce a homomorphism

$$\psi \; : \; \pi_1(M) \to \mathrm{PSL}(2, O_d/\mathcal{P}),$$

with $\phi(\Gamma) = \psi(\pi_1(M))$.

Fix a generating set $\{x_1, \ldots, x_m\}$ for $\pi_1(M)$, with x_i of order p_i. Since p_i is prime $\psi(x_i)$ is trivial or order p_i. We claim $\psi(x_i) = 1$. This will complete the proof in this case, since if $\psi(x_i) = 1$ for each $i = 1, \ldots, m$, then $\psi(\pi_1(M)) = 1$, which implies $\phi(\Gamma) = 1$ and so $\Gamma < \Gamma(\mathcal{P})$. However, these congruence subgroups always have at least two cusps—the parabolic fixed points 0 and ∞ being inequivalent under the action of $\Gamma(\mathcal{P})$. But $M \setminus K$ has only 1 cusp, and so we get a contradiction.

Thus it remains to rule out the image having order p_i. This will follow once we establish that the order of cyclic subgroups of $\mathrm{PSL}(2, O_d/\mathcal{P})$ in our setting is bounded.

From Lemma 4.2, $|x|$ is bounded, and hence the norm of the prime ideal $\mathcal{P}$ is bounded. By definition, this means the field $O_d/\mathcal{P}$ has bounded cardinality. This in turn then bounds the order of the finite group $\mathrm{PSL}(2, O_d/\mathcal{P})$ and hence the order of the cyclic subgroups.

We need to sharpen this slightly. From Lemma 4.2 and the fact that the (field) norm of a quadratic imaginary integer x is simply $|x|^2$, we see that any prime ideal $\mathcal{P}| < x >$ has norm at most 36. Thus the order of the field $O_d/\mathcal{P}$ is bounded by 36. As discussed in §3.2 if $\mathbf{F}$ is a field with ℓ^n (ℓ a rational prime) elements the order of an element in the finite group $\mathrm{PSL}(2, \mathbf{F})$ is either ℓ or at most $\frac{\ell^n+1}{2}$. This bounds the order of cyclic groups at 31. □

Remarks: 1. The hypothesis in Theorem 4.3 cannot be removed completely. For, as we indicate in §6, $\mathbf{RP}^3\#\mathbf{RP}^3$, $\mathbf{RP}^3\#(S^2 \times S^1)$, $\mathbf{RP}^3\#L(4,1)$, and $L(4,1)\#L(4,1)$ all contain an arithmetic knot.

2. The p_i's in the statement of Theorem 4.3 need not be prime, just that all divisors are primes given by the restrictions in Theorem 4.3.

3. A similar argument can be made to show many Seifert fibered spaces over S^2 which are $\mathbf{Z}_2$ rational homology 3-spheres do not contain an arithmetic knot. One simply replaces the conditions on the orders of the cyclic groups appearing in Theorem 4.3 with appropriate conditions on the exceptional fibers.

5 The arithmetic number of a 3-manifold

Let M be a closed orientable 3-manifold, and define *the arithmetic number of M*, denoted $\mathcal{A}(M)$, to be the minimal number of components of a non-empty arithmetic link in M. As remarked in §1, M contains an arithmetic link, $\mathcal{A}(M)$ is finite. Understanding the complexity of $\mathcal{A}(M)$ seems quite challenging as we now discuss.

5.1

We begin by discussing spherical 3-manifolds.

Example 1: A Lens Space L is surgery on the Whitehead link, so that $\mathcal{A}(L) \leq 2$. Theorem 1.2 therefore shows $\mathcal{A}(L) = 2$ for L with $\pi_1(L)$ odd order $\neq 5$.

Example 2: The Poincare homology sphere Σ contains a 2-component arithmetic link (see [7], Example D), so $\mathcal{A}(\Sigma) \leq 2$. This prompts:

Question: Does the Poincare homology sphere contain an arithmetic knot?

Example 3: As discussed in the proof of Theorem 1.2, the sister of the figure eight knot has filling a spherical 3-manifold X with fundamental group $T \times \mathbf{Z}_5$, with T the binary tetrahedral group. Thus $\mathcal{A}(X) = 1$. If we consider the spherical 3-manifold with fundamental group T, this has a surgery description on the Whitehead link (it is a surgery on the trefoil, which in turn is a surgery on the Whitehead link).

In fact these small arithmetic numbers for spherical 3-manifolds is no accident.

Theorem 5.1 *Let M be a 3-manifold finitely covered by S^3. Then $\mathcal{A}(M) \leq 4$.*

Proof: Any such M is orientable, and Seifert fibered. We have already dealt with the case of Lens Spaces, so we assume that M is spherical and not a Lens Space. Any such manifold admits a Seifert fibration, with base S^2 and 3 exceptional fibers. Removing the exceptional fibers produces $D \times S^1$, where D is a twice punctured disc. This is homeomorphic to the complement of the connect sum of two Hopf links. By removing an additional component, we can then convert this link to the 4-chain link 8^4_2 whose complement is arithmetic, being commensurable with Q_1; see [17] for example. □

Motivated by this, we ask:

Question: Do there exist spherical 3-manifolds with arithmetic numbers 3 and 4?

In fact, at present we have no example of *any* closed 3-manifold with $\mathcal{A} > 2$.

5.2

One can say a little more regarding upper bounds for $\mathcal{A}(M)$ for some other classes of manifolds.

Theorem 5.2 *Let M be a connect sum of Lens Spaces L_i, $i = 1, \ldots, m$. Then $\mathcal{A}(M) \leq 2m + 1$.*

Sketch Proof: Let C_k denote the k-component alternating chain link in S^3. Now C_k is arithmetic for only finitely many values of k (see [17]). However, we can remove an additional unknotted component to obtain a link whose complement is commensurable with that of the Whitehead link complement. Briefly, we can arrange the chain link C_k to lie in a standardly embedded solid torus $V_k \subset S^3$. Let γ_k be a meridian of V_k, and define $D_k = C_k \cup \gamma_k$. Then $S^3 \setminus D_k$ is a cover of the Whitehead link complement (see [17] for more details).

Now observe that M can be obtained by a sequence of surgeries on C_{2m}. This proves the result. □

Remark: $2m+1$ is not best possible. For example the connect sum of any two Lens Spaces has a surgery description on the Borromean rings, and this link is arithmetic (see [7]). Thus the arithmetic number of such a manifold is at most 3.

Spherical 3-manifolds all have Heegaard genus at most 2, and manifolds as in Theorem 5.2 have Heegaard genus m. Motivated by this we ask:

Question: Is there a relation between $\mathcal{A}(M)$ and the Heegaard genus of M?

5.3

As is apparent, all our methods are restricted to the case when M is non-hyperbolic. At present we cannot prove $\mathcal{A}(M) > 1$ for *any* closed hyperbolic manifold. We suspect that this indeed so. In fact we suspect that there exists a sequence of closed hyperbolic 3-manifolds $\{M_i\}$ with $\mathcal{A}(M_i) \to \infty$.

5.4

We conclude this section with some additional questions:

Questions: 1. Which closed orientable 3-manifolds contain an arithmetic knot?

2. Classify the spherical 3-manifolds containing an arithmetic knot.

3. Let M be a closed orientable 3-manifold, does M contain at most finitely many distinct arithmetic knots?

4. The arithmetic number of a 3-manifold makes sense in the setting of manifolds with only toroidal boundary components. What can one say in this setting?

6 Final Remarks

As remarked in §1, little seems known about manifolds that do contain an arithmetic knot. In Table 1 below we list some examples from the census of cusped manifolds through 7 tetrahedra given by Snap Pea (see also the exact version Snap and the discussion of this program in [9]). The nomenclature in column 1 being that of the census, and d in column 3 stands for the appropriate $\mathbf{Q}(\sqrt{-d})$ which is the invariant trace-field. Of the manifolds listed in Table 1, $m003$, $m004$, $m206$, $m207$, $s784$, $s958$, $s960$ and $s961$ are derived from a quaternion algebra, the rest are not.

In the table there are descriptions as knot complements in manifolds M_i for $i = 1, \ldots 4$. M_1 is a manifold of odd D-type whose fundamental group has order 24, M_2 is a manifold of even D-type whose fundamental group has order 24, M_3 is a manifold of odd D-type whose fundamental group has order 40, and M_4 is a manifold of Q-type whose fundamental group has order 40 (for terminology see [6] for instance). The manifold Q is the spherical space form with fundamental group the quaternion group of order 8. We have been unable to identify the manifolds, $s956$ and $v2787$. Also some of the Lens Spaces are not determined up to homeomorphism, since we only have information about the fundamental group. This is indicated by use of $*$ in the description of the standard form for the Lens Space. Note that unlike the case of S^3, there are at least two arithmetic knots in $S^2 \times S^1$, $\mathbf{RP}^3 \# \mathbf{RP}^3$, $\mathbf{RP}^3 \# (S^2 \times S^1)$, $\mathbf{RP}^3 \# L(4,1)$, $L(4,1) \# L(4,1)$ and M_1.

Many hyperbolic 3-manifolds contain arithmetic knots, for example those obtained by Dehn surgery on the figure-eight knot. For some other examples of arithmetic knots in closed 3-manifolds, see [5] and [7].

One final remark on the table, in certain cases we have implicitly assumed the Poincare Conjecture and the Spherical Space Form Conjecture in passing from fundamental groups given by Snap Pea to a statement of what the manifolds are.

Table 1 - 1-cusped arithmetic hyperbolic 3-manifolds

Census Description	Volume	d	Description
$m003$	2.0298832128	3	Knot in $L(5,1)$
$m004$	2.0298832128	3	Figure eight knot
$m009$	2.666744783	7	Knot in $\mathbf{RP}^3$
$m010$	2.666744783	7	Knot in $L(6,1)$
$m130$	3.6638623767	1	Knot in $L(16,*)$
$m135$	3.6638623767	1	Knot in $\mathbf{RP}^3\#L(4,1)$
$m136$	3.6638623767	1	Knot in $\mathbf{RP}^3\#\mathbf{RP}^3$
$m139$	3.6638623767	1	Knot in $L(24,*)$
$m140$	3.6638623767	1	Knot in $L(4,1)$
$m206$	4.059766426	3	Knot in $L(5,3)$
$m207$	4.059766426	3	Knot in $L(3,1)\#L(3,1)$
$m208$	4.059766426	3	Knot in $L(20,*)$
$s118$	4.059766426	3	Knot in $L(8,*)$
$s119$	4.059766426	3	Knot in Q
$m410$	5.074708032	3	Knot in $\mathbf{RP}^3\#(S^2 \times S^1)$
$s594$	5.074708032	3	Knot in $\mathbf{RP}^3\#L(8,*)$
$s595$	5.074708032	3	Knot in $\mathbf{RP}^3\#L(4,1)$
$s772$	5.3334895669	7	Knot in $L(24,*)$
$s773$	5.3334895669	7	Knot in $\mathbf{RP}^3\#(S^2 \times S^1)$
$s775$	5.3334895669	7	Knot in M_1
$s777$	5.3334895669	7	Knot in $S^2 \times S^1$
$s778$	5.3334895669	7	Knot in $L(24,*)$
$s779$	5.3334895669	7	Knot in M_1
$s781$	5.3334895669	7	Knot in $L(4,1)\#L(4,1)$
$s784$	5.3334895669	7	Knot in $L(3,1)\#(S^2 \times S^1)$
$s786$	5.3334895669	7	Knot in $L(4,1)\#L(4,1)$
$s787$	5.3334895669	7	Knot in M_2
$v1858$	5.497935651	1	Knot in $\mathbf{RP}^3\#L(4,1)$
$v1859$	5.497935651	1	Knot in $\mathbf{RP}^3\#L(8,*)$
$v2787$	6.02304602	2	Not identified
$v2789$	6.02304602	2	Knot in $\mathbf{RP}^3\#L(8,*)$
$s955$	6.089649638	3	Knot in $L(40,*)$
$s956$	6.089649638	3	Not identified
$s957$	6.089649638	3	Knot in M_3
$s958$	6.089649638	3	Knot in $L(4,1)\#L(3,1)$
$s960$	6.089649638	3	Knot in M_4
$s961$	6.089649638	3	3-fold cyclic cover of $m004$
$v2873$	6.089649638	3	Knot in $S^2 \times S^1$
$v2874$	6.089649638	3	Knot in $\mathbf{RP}^3\#\mathbf{RP}^3$

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