373K Algebra I, Homework 5

From Artin

Chapter 2 (p. 74): 11.3, 11.5, 11.6, 11.9.

Others:

1. Let G be a group and $S \subset G$ (only a subset, possibly empty). Let $< S >= \cap\{H < G : S \subset H\}$.

(i) Prove that $< S >$ is a subgroup of G, called the *subgroup generated by* S.

(ii) Let G denote the dihedral group D_n, ρ a rotation of order n, and σ any reflection. Prove that $< \{\rho, \sigma\} >= G$.

2. Referring to Question **1** above. Let G be a group and $S = \{aba^{-1}b^{-1}\}$. The group $< S >$ is called the *commutator subgroup* of G and denote G'.

(i) Prove that G' is a normal subgroup of G.

(ii) Prove that G/G' is abelian.

(iii) Prove that if N is a normal subgroup of G and G/N is abelian then $G' < N$.

3. Referring to Question 2 above. What is D_4/D_4'?

4.(i) Compute the orders of $\mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z})$ and $\mathrm{PSL}(2, \mathbf{Z}/5\mathbf{Z})$. Prove that $\mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z})$ has no subgroup of order 6.

(ii) Prove that $\mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z})$ is not simple but $\mathrm{PSL}(2, \mathbf{Z}/5\mathbf{Z})$ is simple.

5. Let G be a finite group, let p be a prime, and let $H < G$ be a normal subgroup. Prove that if H and G/H have orders which are a power of p, then $|G|$ is a power of p.

6. Let $G = S_4$ and $H =< 1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (2\ 3)(1\ 4) >$. Prove that $H < G$ is normal and identify G/H.

Sample Midterm 1 Questions

1. Answer the following *true* or *false*. You must explain your answers to get full credit.

(a) There is a non-abelian group of order 17.

(b) There is a non-abelian group of order 14.

(c) There is an abelian non cyclic group of order 49.

(d) S_5 contains an element of order 6.

2. Suppose A is a normal subgroup of G and $H < G$ and these satisfy:

(i) A is abelian,

(ii) $A.H = G$.

Show that $A \cap H$ is a normal subgroup of G.

(**Hint**: Show $A \subset N_G(A \cap H)$ and $H \subset N_G(A \cap H)$.)

3.(a) Let G be a group containing normal subgroups H, K such that $H \cap K = 1$ and $G = HK$. Show that G is isomorphic to $H \times K$ (**Hint:** Consider G/K and G/H).

(b) Use (a) to deduce that an Abelian group of order 9 is cyclic or isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$.

4. Prove that if $\alpha \in S_n$ is an m-cycle (i.e has the form $(a_1\ a_2\ \dots\ a_m)$ for distinct integers $a_1, \dots a_m \in \{1, \dots n\}$) then α is a product of transpositions (ie. permutations of the form $(i\ j)$ for some $1 \leq i \neq j \leq n$).

5. Suppose that G is a group and $N < G$ a normal subgroup. Assume that there is no normal subgroup $M < G$ with $N < M$. Prove that G/N is simple.