SAGE warm-up lectures : Berni

What are complex manifolds?

(1) Locally look like $\mathbb{C}^n$ instead of $\mathbb{R}^n$

top space w/ coord charts $\{U_i, \varphi_i\}$ s.t. $\varphi_i : U_i \to \mathbb{C}^n$

& $\varphi_i \circ (\varphi_j)^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ holomorphic

Given a cmplx mfld M looks like $\mathbb{C}^n$ locally. $\dim_{\mathbb{C}} M = n$

(2) Ordinary real mfld w/ extra tensor on it. $\dim_{\mathbb{R}} M = 2n$.

M diff mfld w/ $\underset{\text{cmplx structure}}{\underline{J}}$ (1,1)-tensor & $J^2 = -1$.

$J : V \mapsto W$ restrict to $x \in M$, $J_x : T_x M \to T_x M$

M is an almost complex mfld.

Nijenhuis tensor defined in terms of $J = 0 \iff M$ is a cmplx mfld.

Note that for $x \in M$, $J_x : T_{x,\mathbb{R}} M \to T_{x,\mathbb{R}} M$ s.t. $J_x^2 = -1$

$T_{x,\mathbb{R}} M \longleftrightarrow \mathbb{C}^n$, J_x & i coincide.

Thinking of $T_{x,\mathbb{R}} M$ as $\mathbb{C}^n$ & thinking of $\mathbb{C}^n$ as vector space $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$

means we can think of $T_{x,\mathbb{R}} M$ as $\mathbb{R}\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$

COMPLEXIFY!!

Replace $\mathbb{R}$ w/ $\mathbb{C}$, let $T_x M \equiv \mathbb{C}\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$

$$= \mathbb{C}\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\}$$

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right) \longleftrightarrow \begin{matrix} dz^j = dx^j + i\,dy^j \\ d\bar{z}^j = dx^j - i\,dy^j \end{matrix}$$

$T_x^{1,0}M = \mathbb{C}\{\frac{\partial}{\partial z^i}\}$ Holomorphic tangent space

$T_x^{0,1}M = \mathbb{C}\{\frac{\partial}{\partial \bar{z}^i}\}$ Antiholomorphic tangent space

$\dim_{\mathbb{R}} T_x^{1,0}M = T_x^{0,1}M = n$

$J^2 = -1$ has evals $\pm i$.

$T_x^{1,0}M = \{\text{eigenspace of } i\}$

$T_x^{0,1}M = \{\text{eigenspace of } -i\}$

$T_xM = T_x^{1,0}M \oplus T_x^{0,1}M$

<u>DUALIZE</u>!

$T_x^*M = (T_x^*)^{1,0}M \oplus (T_x^*)^{0,1}M$

$\Lambda^k(T_x^*M) = \bigoplus_{p+q=k} (\Lambda^p T_x^{*(1,0)}M \otimes \Lambda^q T_x^{*(0,1)}M)$

$\Lambda^k(M) = \bigoplus_{p+q=k} \Lambda^{p,q}(M)$

Choose holomorphic coords $(z^1, .. z^n)$
$\alpha \in \Lambda^{p,q}(M)$

$\alpha = \sum \alpha_{IJ}\, dz_I \wedge d\bar{z}_J$
$dz_I = dz_{i_1} \wedge .. dz_{i_p}$ if $I = \{i_1, .., i_p\}$

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d raises degree by 1

$d : \Lambda^k(M) \to \Lambda^{k+1}(M)$
$\Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)$

$d = \partial + \bar\partial$

$\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)$

$\bar\partial : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$

$$\left.\begin{array}{l} \partial^2 = \bar\partial^2 = 0. \\ \partial\bar\partial = -\bar\partial\partial \end{array}\right\} \Leftarrow d^2 = 0$$

Metrics

M complex mfld.

a hermitian metric on M is a smoothly varying, pos-def, herm IP on each hol tangent sp $T_x^{1,0}M$. So if V, W are hol vector fields & $\alpha, \beta \in \mathbb{C}$, then

$$h(\alpha V, \beta W) = \alpha h(V, \beta W) = \bar\beta h(\alpha V, W)$$

$\Rightarrow$ particular connection compatible w/ h.

Let $\{\varphi_i\}$ be unitary coframe (1-forms) s.t. $h = \sum \varphi_i \otimes \bar\varphi_i$

Let $\varphi_j = \alpha_j + i\beta_j$

Then $h = \sum(\alpha_j + i\beta_j) \otimes (\alpha_j - i\beta_j)$

$$= \sum \underbrace{(\alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j)}_{\uparrow \text{ symmetric}} + i \sum \underbrace{(\beta_j \otimes \alpha_j - \alpha_j \otimes \beta_j)}_{\uparrow \text{ alternating}}$$

$T_{x,\mathbb{R}} M \cong T_x^{1,0} M$

So we can write $h = \operatorname{Re} h + i \operatorname{Im} h$

$$= \underset{\substack{\uparrow \\ \text{Riemannian metric}}}{g} - 2i\omega$$

ω: associated (1,1) form

$d\omega = 0 \Rightarrow$ metric is Kähler. (so they are complex, Riemannian, symplectic)

$g(X,Y) = \omega(JX, Y)$

Equivalent conditions for a Kähler mfld

1) $d\omega = 0$
2) The metric connection is torsion-free.
3) h Osculates to order 2 to the Euclidean metric

$$h = \sum (\delta_{ij} + \eta_{ij})\, dz^i \otimes d\bar{z}^j \quad \text{at } p$$

η_{ij} vanishes to 2^{nd} order at p.

Hodge Decomposition

Let M cmpt Riem mfld. The metric g induces an inner product on differential forms. $g \Rightarrow T_x M \cong T_x^* M$.

1-forms dual to v, w: $\langle \theta, \tau \rangle_x = g(v,w)(x)$

$\Lambda^k(T_x^* M)$. Let $e^1, \dots e^n$ coframe for g at x.

$$g = (e^1)^2 + \dots (e^n)^2$$

$\{e^{i_1}(x) \wedge \dots e^{i_k}(x)\}$ is ON basis for Λ^k.

Globalize by letting $dvol = e^1 \wedge \dots \wedge e^n$

Inner product on $\Lambda^k(M)$:

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_x \, dvol$$

For d, we have a formal adjoint d^*

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

Laplacian:

$$\Delta = dd^* + d^*d$$

Harmonic forms:

$$\mathcal{H}(M) = \ker \Delta$$

Hodge theorem: $\mathcal{H}^k(M) \cong H^k(M)$.

Kähler mfld

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$$

$$\mathcal{H}^{p,q}(M) = \overline{\mathcal{H}^{q,p}(M)}$$

$$H^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$$

$$\left.\begin{aligned} H^k(M, \mathbb{C}) &= \bigoplus H^{p,q}(M) \\ H^{p,q}(M) &= \overline{H^{q,p}(M)} \end{aligned}\right\} \text{Hodge decomposition}$$