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- Holomorphic Bundles & Chern Classes

$$\begin{array}{c} E \\ \pi \downarrow \\ M \end{array}$$ smooth Cx vect bundle over M

We say E is holomorphic if E is a Cx mfld, π is a holo map, and $\pi^{-1}(p)$ is biholomorphic to $\mathbb{C}^r$ $\forall p \in M$.

or transition functions $g_{ab}\ U_{ab} \longrightarrow GL(r,\mathbb{C}) \hookrightarrow M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ are holomorphic

- Chern Classes [$\exists$ other viewpoints, but well use curvature]

Let $\{e_i\}$ be a frame $\quad \nabla e_i = \sum_{j=1}^{r} \theta_{ij} \otimes e_j$; $\Theta = d\theta - \theta \wedge \theta$

connection of 1-form

so $\nabla^2 e_i = \sum \Theta_{ij} \otimes e_j$ under a change of frame

$\Theta \longmapsto A \Theta A^{-1}$ for A the change of frame

If f is a homogeneous polynomial of degree i, in the entries of a matrix. $f: M_r(\mathbb{C}) \longrightarrow \mathbb{C}$, it is <u>invariant</u> if $f(APA^{-1}) = f(P)$, $P \in M_r(\mathbb{C})$, $A \in GL(r,\mathbb{C})$.

eg $\det(P + tI) = f_n(P) + f_{n-1}(P)t + \cdots + f_1(P)t^{n-1} + t^n$.

f_i are the elementary symmetric functions of the e-vals of P

$f_1 = \mathrm{Tr}$, $f_n = \det$.

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$(E,\nabla) \longrightarrow N$. a v.b. w connection

$f_i\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)$ is a $2i$-form. we call it

the ith Chern form $c_i(E,\nabla)$

- Now $c_i(E,\nabla)$ is closed, and $[c_i(E,\nabla)] \in H^{2i}(N)$ is independent of ∇.
- we will call call this class $C_i(E)$, and the Chern class of a manifold is the Chern class of its tangent bundle

· Preview: there will be a tripartite division into pos, neg, 0 chern classes.

- Kähler-Einstein metrics

recall $\omega(X,Y) = g(JX,Y)$

$$g_{\mathbb{C}} = g_{\alpha\bar\beta}\,(dz^\alpha \otimes d\bar z^\beta + d\bar z^\beta \otimes dz^\alpha)$$

$$\Rightarrow \quad \omega = \sqrt{-1}\sum g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$ [Beware sign & constant conventions]

$$R_{\alpha\bar\beta\gamma\bar\delta} = -\frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta} g_{\gamma\bar\delta} + g^{\lambda\bar\mu}\frac{\partial}{\partial z^\alpha} g_{\gamma\bar\mu}\frac{\partial}{\partial \bar z^\beta} g_{\lambda\bar\delta}$$

$$Ric_{\alpha\bar\beta} = R^{\delta}_{\delta\bar\beta\alpha} = R_{\alpha\bar\beta} = \frac{-\partial^2}{\partial z^\alpha \partial \bar z^\beta}\log\det(g_{\gamma\bar\delta})$$

Ricci form: $\rho(X,Y) := \frac{1}{2}Rc(JX,Y)$; $\rho = \sqrt{-1}\,R_{\alpha\bar\beta}\,dz^\alpha \wedge d\bar z^\beta$

Holomorphic Bisectional Curvature.

$$K(Z,W) = Rm(Z,\bar{Z},W,\bar{W}) / |Z|^2|W|^2.$$

Rk. this depends on a 4 $\mathbb{R}$-dimensional thing, no analog in Riemannian world

Uniformization result.

(M,g) complete Kähler mfld, g constant (hol) bisectional curvature

$R_{i\bar{j}k\bar{l}} = \lambda(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$. Then $\tilde{M}$ ← universal cover is

$\mathbb{CP}^n$, $\mathbb{C}^n$, or $\mathbb{B}^n$ and up to scaling

$\lambda = 1$ $\quad \mathbb{CP}^n \quad g_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log(1+|z|^2)$, $Ric_{i\bar{j}} = (n+1)g_{i\bar{j}}$

$\lambda = 0$ $\quad$ flat $\mathbb{C}^n$

$\lambda = -1$ $\quad \mathbb{B}^n \; \omega_g = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1-|z|^2)$.

- Kähler - Einstein $\rho = \lambda\omega$, $\lambda \in \mathbb{R}$

$c_1 = 0$: If (M^n, g_0) is closed Kähler mfld, $c_1(M) = 0$, then $\exists$ Kähler metric g, $[\omega] = [\omega_0]$, $Rc(g) \equiv 0$, ie Kähler-Ricci-flat ie Calabi-Yau (?)

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$c_1 < 0$: M^n closed cx $c_1(M) < 0$. Then $\exists$ Kähler-Einstein metric g on M unique up to homothety (scaling) of neg. scalar curvature

$c_1 > 0$: obstructions to existence of K-E metrics

eg Futaki - Invariant.

M^n closed, $c_1 > 0$. fix g so $[\omega] = c_1(M)$.

then $\exists$ smooth $f : M \to \mathbb{R}$. st

$$\rho - 2\pi\omega = \sqrt{-1}\,\partial\bar\partial f, \quad \left(\text{unique if normalized as } \int_M e^{-f} d\mu = 1\right)$$

Let $\eta(M)$ be the space of (real) holo vector fields on M

$$\mathcal{F}_{[\omega]}(V) := \int_M V(f)\, d\mathrm{vol} \quad \text{depends only on } [\omega]$$

$$\text{K-E} \Rightarrow \mathcal{F}_{[\omega]} = 0. \quad \text{However} \quad 0 = \mathcal{F}_{[\omega]} \not\Rightarrow \text{K-E}.$$

However, if one exists its unique up to scaling and pullback by biholomorphism of M

Frankel Conj. Compact, n-dim Kähler. Pos Bisectional curv. are Biholomorphic to $\mathbb{CP}^n$

For Surfaces: $c_1 > 0$, and Lie alg of automorphism group reductive $\Rightarrow \exists$ K-E metric w pos. scalar curvature

Solitons. $Rc + \lambda g + \frac{1}{2}\mathcal{L}_X g = 0 \quad \lambda \in \mathbb{R}.$

X a real v.f., infinitesimal auto. of J

$\Updownarrow$

(1,0) part holomorphic

Gradient Soliton. $X = \nabla f$, some real f

K-E metrics and non trivial gradient solitons are mutually exclusive

Pf $\quad X = \text{grad } f \Rightarrow F_{[\omega]}(X) = \int_M \langle X, X\rangle \,d\mathrm{vol} = \|X\|^2 > 0 \quad \square$

Solitons are unique on cpct Kähler-mflds up to holomorphic automorphisms