A Crash Course on Compact Complex Surfaces

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Abstract

- “Analytic invariants” of complex manifolds that the generalizations of the genus of curves, and their birationally invariant nature.
- Blow-up of a surface at a point.
- Birational classification of complex surfaces via minimal models. Enriques-Kodaira classification.
- Canonical models.
- Calabi-Yau manifolds and $K3$ surfaces.
- Fano manifolds and del Pezzo surfaces.

Examples of Compact Complex Surfaces

1. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ ($\cong$ smooth quadric surface in $\mathbb{P}^3$), smooth hypersurfaces in $\mathbb{P}^3$, two-dimensional submanifolds of $\mathbb{P}^n$, Cartesian products of two compact Riemann surfaces.
2. *fake projective planes* $:=$ compact complex surfaces with $b_1 = 0$, $b_2 = 1$ not isomorphic to $\mathbb{P}^2$. Such a surface is projective algebraic and it is the quotient of the open unit ball in $\mathbb{C}^2$ by a discrete subgroup of PU$(2, 1)$.The first example (Mumford surface) was constructed Mumford using p-adic tecnhniques. Recently, all possible (17 known finite classes plus four possible candidates and no more) fake projective planes have been enumerated by Gopal Prasad and Sai-Kee Yeung. See abstract for colloquium on March 26, 2007.
3. *Ruled surface* $:=$ $\mathbb{P}^1$-bundle over a compact Riemann surface.
   - Can be shown: All ruled surfaces are projectivizations of rank-two vector bundles over compact Riemann surfaces.
   - Hirzebruch surfaces: $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, $n = 0, 1, 2, \ldots$
4. *Elliptic surface* $:=$ total space of a holomorphic fibration over a compact Riemann surface with generic fiber being a smooth elliptic curve.
5. 2-dimensional complex tori: C^2/Λ, where $\Lambda \cong \mathbb{Z}^4$ is a discrete lattice in $\mathbb{C}^2$.
6. *Hopf surface* $:=$ compact complex surface with universal cover $\mathbb{C}^2 - \{0\}$. For example, $\left(\mathbb{C}^2 - \{0\}\right)/\mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{C}^2$ is generated by $\mathbb{C}^2 \longrightarrow \mathbb{C}^2 : z \mapsto 2\,z$. (The Hopf surface is compact and non-Kähler.)

In the Beginning ...

Goddess

Said

Let there be ...

CURVES.

I am not joking; ask the string theorists.

Classification of smooth compact complex curves by genus

analytic/topological genus $g(C) = h^0(C, \Omega^1_C) = h^0(C, K_C)$

degree of canonical bundle $\deg(K_C) = 2g(C) - 2$

$g(C) = h^0(K_C)$	$\widetilde{C}$	curvature	$\deg(K_C)$	kod(C)
0	$\mathbb{P}^1$	positive	< 0	$-\infty$
1	$\mathbb{C}$	flat	$= 0$	0
≥ 2	$\mathbb{CH}^1$	negative	> 0	1

Canonical maps for curves of genus ≥ 2

For a a line bundle L on a smooth compact complex curve (Riemann surface) C,

$$\deg(L) = \int_C c_1(L) = \left\{ \begin{array}{c} \text{sum of orders of zeros and poles of} \\ \text{a generic meromorphic section of } L \end{array} \right\}$$

$$h^0(C,L) > 0 \implies \deg(L) \geq 0 \,; \quad \text{conversely} \quad h^0(C,L) = 0 \impliedby \deg(L) < 0$$

The canonical map ϕ_{K_C} for a curve C of genus $g \geq 2$ is

$$\begin{array}{ccc} C & \overset{\phi_{K_C}}{\dashrightarrow} & \mathbb{P}(H^0(C,K_C)) \cong \mathbb{P}^{g-1} \\ x & \longmapsto & [s_0(x) : \cdots : s_{g-1}(x)] \end{array}$$

where $s_0, \ldots, s_{g-1}$ form a basis for $H^0(C, K_C) \cong \mathbb{C}^g$.

ϕ_{K_C} is undefined at each $x \in C$ with $s_0(x) = \cdots = s_{g-1}(x) = 0$. Such points of C are called base points of ϕ_{K_C}.

Note: $\quad g \leq 0 \implies \deg(K_C) = 2(g-1) < 0 \implies$ no canonical map.

Canonical images of curves of genus ≥ 1

FACT: For C with $g(C) = h^0(C, K_C) \geq 1$, ϕ_{K_C} is in fact base-point-free and is therefore a holomorphic map $C \longrightarrow \mathbb{P}(H^0(C, K_C)) \cong \mathbb{C}^{g-1}$.

For C with $g(C) \geq 1$,

$$\text{kod}(C) := \text{dimension of image of canonical map } \phi_{K_C}.$$

$g(C)$	kod(C)	ϕ_{K_C}	type
0	$-\infty$		$C \cong \mathbb{P}^1$ Riemann sphere
1	0	constant	$C \cong \mathbb{C}/\Lambda$ complex torus
2	1	$C \xrightarrow{2:1} \mathbb{P}^1$	hyperelliptic [a]
≥ 3	1	$C \xrightarrow{2:1} \mathbb{P}^{g-1}$	hyperelliptic
		$C \hookrightarrow \mathbb{P}^{g-1}$	projective curve of degree $2g-2$

[a] hyperelliptic curve := $(2 : 1)$-branched cover of $\mathbb{P}^1$

Discrete invariants generalizing genera of curves

For a curve C,

$$g(C) :=: \text{topological genus} \;=\; \text{analytic genus} := h^{1,0}(C)$$

$$h^{1,0}(C) = h^{n,0}(C) = h^0(C, \Omega^n(C)) = h^0(C, K_C), \quad n := \dim_{\mathbb{C}}(C) = 1.$$

For a complex manifold M of dimension n,

$$\begin{aligned}
\text{irregularity, } q \quad &:= \quad h^1(M, \mathcal{O}_M) = h^{0,1}(M) \overset{\text{Kähler}}{=} h^{1,0}(M) \\
\text{geometric genus, } p_g \quad &:= \quad h^{n,0}(M) = h^0(M, \Omega^n_M) = h^0(M, K_M) \\
\text{Hodge Numbers, } h^{p,0}(M) \quad &:= \quad h^0(M, \Omega^p_M), \;\; 0 \le p \le n \\
m^{\text{th}} \text{ plurigenus, } P_m(M) \quad &:= \quad h^0(M, K_M^m), \;\; m = 1, 2, 3, \ldots
\end{aligned}$$

However, these “biholomorphic” invariants are in fact only “birational” (or bimeromorphic) invariants due to the operation of blow-up of points.

Blow-up of $\mathbb{C}^2$ at the origin

$$\begin{array}{ccl} \mathsf{Bl}_{\mathbf{0}}(\mathbb{C}^2) & := & \{ \, (\, (x,y), [u:v] \,) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xv = yu \, \} \\ \pi \downarrow & & \quad \downarrow \\ \mathbb{C}^2 & \in & (x,y) \end{array}$$

- $(\,(x,y),[u:v]\,) \in \mathsf{Bl}_{\mathbf{0}}(\mathbb{C}^2) \iff$ $(x,y) \in \mathbb{C}^2$ lies in the 1-dimensional subspace $[u:v] \in \mathbb{P}^1$.
- $\pi^{-1}(\mathbf{x}) = (\,\mathbf{x}, [\mathbf{x}]\,)$, for any $\mathbf{x} \neq \mathbf{0}$.
- $\pi^{-1}(\mathbf{0}) = \mathbf{0} \times \mathbb{P}^1 \cong \mathbb{P}^1 =$ space of tangential directions through $\mathbf{0} \in \mathbb{C}^2$.
- $\mathsf{Bl}_{\mathbf{0}}(\mathbb{C}^2) - \pi^{-1}(\mathbf{0}) \xrightarrow{\pi} \mathbb{C}^2 - \mathbf{0}$ is a biholomorphism.
- Both $\mathsf{Bl}_{\mathbf{0}}(\mathbb{C}^2)$ and $\mathbb{C}^2$ smooth, i.e. blow-ups and blow-downs of points preserve smoothness.
- Blow-up increases the second Betti number by 1. Blow-down decreases it by 1.

Hartogs' Extension Theorem and Birational Invariants

DEFINITION A *birational* (or *bimeromorphic*) map $X \dashrightarrow Y$ between two complex manifolds is a biholomorphic map $X \backslash S \longrightarrow Y \backslash T$, where S, T are proper subvarieties of X and Y respectively.

In the complex algebraic geometry setting, the various "biholomorphic invariants" (defined via differential forms) introduced above are in fact "birational invariants":

1. Hartog's Extension Theorem: meromorphic ($\mathbb{C}$-valued) functions whose loci of non-holomorphicity has codimension ≥ 2 extend to holomorphic functions.
2. The existence of birational but non-biholomorphic maps in dimensions ≥ 2. Prototypical examples: blow-up of points.
3. The locus of indeterminacy of a birational map between algebraic manifolds has codimension at least two.

Consequently, pull-backs of holomorphic forms of algebraic manifolds under birational maps have loci of indeterminacy of codimension ≥ 2; hence such pull-backs always extend to holomorphic forms by Hartogs' Theorem.

PROBLEM: Each birational equivalence class of surfaces contains infinitely many smooth surfaces.

Q: Is there a distinguished representative in a given birational class?

Intersection numbers of two curves on a surface

Let $L = [D]$ be a holomorphic line bundle defined by the divsior D on a compact complex surface X. Let $C \subset X$ be a curve embedded in X. Then, the intersection number of C and D is definedy by

$$D \cdot C := \int_C c_1([D]|_C) = \int_C c_1(L|_C) = \deg(L|_C) = \left\{ \begin{array}{l} \text{sum of orders of zeros and poles of a} \\ \text{generic meromorphic section of } L|_C \end{array} \right\}$$

In particular, $D \cdot C < 0 \implies$ the restriction of $L = [D]$ to C has no nonzero sections.

FACT: The exceptional locus $E := \pi^{-1}(\mathbf{0}) \cong \mathbb{P}^1$ is a copy of $\mathbb{P}^1$ embedded in the surface $X := \mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^2)$ with self-intersection -1;

equivalently, $[E]|_E = N_{E/X}$ is equivalent to the tautological bundle over $\mathbb{P}^1$;

equivalently, $\deg([E]|_E) = -1$.

Consequently, $N_{E/X}$ has no holomorphic sections and E cannot be holomorphically deformed in X.

Birational maps of complex surfaces & minimal surfaces

FACT: Every birational map between two smooth compact complex surfaces is a finite succession of blow-ups and blow-downs of points

RECALL: The exceptional locus of a blow-up is a copy of $\mathbb{P}^1$ with self-intersection -1.

DEFINITION: An copy of $\mathbb{P}^1$ in a surface with self-intersection -1 is a called a (-1)-curve.

Castelnuovo-Enriques Blow-down Criterion:

An embedded curve C in a surface X can be blown down to a point

$\iff$ C is a (-1)-curve

$\iff$ $C \cdot C < 0$ and $K_X \cdot C < 0$.

A way to find a distinguished representative within a birational class: Given a smooth surface, locate all of its (-1)-curve(s), blow them down one at a time, until we reach smooth surface with no (-1)-curves. This process must terminate in finitely many steps because each blow-down lowers second Betti number by 1.

DEFINITION: A compact complex surface is said to be *minimal* if it contains no (-1)-curves.

Enriques-Kodaira Classification of (minimal) complex surfaces

class	kod(X)	$a(X)$	p_g	q	$K_X{}^? = 0$	b_2	c_1^2	$c_2 = e$
rational		2	0	0		$1, 2$	$8, 9$	$3, 4$
ruled[a] $(g \geq 1)$	$-\infty$	2	0	g		2	$8(1-g)$	$4(1-g)$
VII$_0$		$0, 1$	0	1			$-b_2$	b_2
$K3$		$0, 1, 2$	1	0	1	22	0	24
Enriques		2	0	0	2	10	0	12
2-tori	0	$0, 1, 2$	1	2	1	6	0	0
hyperelliptic		2	0	1	$2, 3, 4, 6$	2	0	0
primary Kodaira		1	1	2	1	4	0	0
2^{nd}-ary Kodaira		1	0	1	$2, 3, 4, 6$	0	0	0
properly elliptic	1	$1, 2$					0	≥ 0
general type	2	2	> 0	$b_1/2$		> 0		> 0

$a(X) :=$ trdeg$_{\mathbb{C}}$ $(\mathbb{C}(X))$, the transcendence degree of the field $\mathbb{C}(X)$ of rational functions of X. $a(X)$ is called the algebraic dimension of X. The following inequalities hold: kod$(X) \leq a(X) \leq \dim_{\mathbb{C}}(X)$. Furthermore, $a(X) = 1 \implies X$ is an elliptic surface.

[a] A *ruled surface* is a $\mathbb{P}^1$-bundle over a curve of genus $g \geq 0$.

The Kodaira Dimension of a Compact Complex Manifold

Let L be a line bundle on a compact connected complex manifold M. Its *Iitaka-Kodaira dimension* is

$$\kappa(M,L) := \begin{cases} -\infty, & \text{if } h^0(M,mL)=0,\ \forall m \geq 1, \\ \sup\left\{ \dim_{\mathbb{C}} \phi_{mL}(M) \,\middle|\, \begin{array}{c} m \in \mathbb{N} \\ h^0(M,mL) > 0 \end{array} \right\}, & \text{otherwise} \end{cases}$$

FACT: $\kappa(M,L) \in \{-\infty, 0, 1, 2, \dots, a(M)\}$, where $a(M) := \text{trdeg}_{\mathbb{C}}(\mathbb{C}(M))$ is the algebraic dimension of M, and $0 \leq a(M) \leq \dim_{\mathbb{C}}(M)$.

FACT: If $\kappa := \kappa(M,L) \geq 0$, then there exists a positive integer m_0 and a constant $C > 0$ such that

$$\frac{1}{C}\, m^{\kappa} \leq h^0(M, L^{\otimes m}) \leq C\, m^{\kappa}.$$

DEFINITION: The *Kodaira dimension* $\text{kod}(M) := \kappa(M, K_M)$.

$\text{kod}(M)$ is the maximal dimension of the images of the pluricanonical maps ϕ_{mK_M} of M.

More on examples of complex surfaces

- Hypersurfaces X in $\mathbb{P}^3$ of degree d. The adjunction formula[a] states $K_X = (K_{\mathbb{P}^3} + [X])|_X = \mathcal{O}_X(d-4)$, since $K_{\mathbb{P}^3} = \mathcal{O}(-4)$.
  - $d > 4 \Longrightarrow K_X$ is ample, $\mathrm{kod}(X) = 2$. X is of general type.
  - $d = 4 \Longrightarrow K_X$ is trivial, $\mathrm{kod}(X) = 0$. Simply connected complex manifolds with trivial canonical bundles are called *Calabi-Yau* manifolds. 2-dimensional Calabi-Yau manifolds are also called $K3$ surfaces.
  - $d < 4 \Longrightarrow -K_X$ is ample, $\mathrm{kod}(X) = -\infty$. Complex manifolds with ample anticanonical bundls are called *Fano* manifolds. 2-dimensional *Fano* manifolds are also called *del Pezzo* surfaces. The only minimal del Pezzo surfaces are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. All other del Pezzo surfaces are “successive blow-ups” of $\mathbb{P}^2$ at $n \in \{1, 2, \ldots, 8\}$ “general” points, hence are not minimal.
- The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces

$$F_n := \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\right), \quad \text{with} \quad n = 0, 2, 3, \ldots$$

  $F_1 =$ blow-up of $\mathbb{P}^2$ at one point $= \mathbb{P}^2 \# \overline{\mathbb{P}^2}$, and is NOT mimimal. $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. A *Hirzebruch surface* is a ruled surface ($\mathbb{P}^1$-bundle) X over $\mathbb{P}^1$. Grothendieck splitting[b] implies

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) =: F_n, \quad \text{for some} \quad n \geq 0.$$

[a] $K_{\mathbb{P}^3} = \bigwedge^n (\Omega^1 \mathbb{P}^3)^* = \wedge^{n-1} (\Omega^1 X)^* \otimes N^*_{X/\mathbb{P}^3} = K_X \otimes [-X]\,|_X$.

[b] Vector bundles over $\mathbb{P}^1$ are direct sums of line bundles.

Minimal models in higher dimensions & nefness of line bundles

Importance of rational curves and nefness of canonical bundle

DEFINITION: A line bundle L on a complex projective manifold M is said to be *nef* if $L \cdot C \geq 0$ for each irreducible curve C in M.

- RECALL (Castelnuovo Criterion): A curve $C \subset X$ if a (-1)-curve if and only if $C \cdot C < 0$ and $K_X \cdot C < 0$. Hence, K_X nef $\Longrightarrow$ X minimal (i.e. no (-1)-curves).
- FACT: For a smooth projective surface X, K_X not nef $\Longrightarrow$ either $X \cong \mathbb{P}^2$, or X is ruled ($\mathbb{P}^1$-bundle over a curve), or X is NOT minimal (i.e. it contains (-1)-curves).
- Thus, if $\mathrm{kod}(X) \geq 0$, then K_X is nef if and only if X is minimal (i.e. no (-1)-curves).
- FACT: If the canonical bundle K_M of a compact complex manifold M is not nef, then, there exists a rational curve C (i.e. C is birationally equivalently to $\mathbb{P}^1$) which is contracted by π to a point and satisfies $K_M \cdot C < 0$. [Contraction Theorem]

DEFINITION: A complex projective manifold is said to be *minimal* if its canonical bundle is nef.

DEFINITION: A line bundle on a compact complex manifold is said to be *nef* if for every $\varepsilon > 0$, there is a smooth hermitian metric h_ε on L such that $\mathbf{i}\,\Theta_{h_\varepsilon(L)} \geq -\varepsilon\omega$, where ω is any fixed Hermitian metric on M.

Differential-geometric Intuition of Nefness

FACT: For a Kähler metric on a complex manifold M, its Ricci curvature (of its underlying Riemannian metric) is the induced curvature on the canonical bundle

$$K_M := \bigwedge^n (T^{1,0}M)^* .$$

Note: $c_1(M) := c_1\left(T^{1,0}M\right) = -c_1\left((T^{1,0}M)^*\right) = -c_1\left(\bigwedge^n (T^{1,0}M)^*\right) = -c_1(K_M)$.

DEFINITION: Intersection of a line bundle L on a compact Kähler manifold (M, ω) with a curve $C \subset M$:

$$L \cdot C := \int_C c_1(L|_C) = \deg(L|_C) = \left\{ \begin{array}{c} \text{sum of orders of zeros and poles of a} \\ \text{generic meromorphic section of } L|_C \end{array} \right\}$$

In particular,

$$K_M \cdot C := \int_C c_1(K_M|_C) = \deg(K_M|_C) = \int_C \left\{ \begin{array}{c} \text{class of curvature of} \\ \text{any metric on } K_M \end{array} \right\} = \int_C \mathrm{Ric}(\omega)$$

So, conditions such as $K_M \cdot C \geq 0$, $K_M \cdot C \leq 0$, etc. are positivity/negativity conditions on "average Ricci curvature of (M, ω) restricted on curves".

Difficulties in Minimal Model Program in higher dimensions

PROBLEM:	The image of a smooth variety under contraction of a rational curve may be singular.
Fix:	Allow minimal models to be (mildly) singular.

PROBLEM:	Singularities arising from "small" contractions (codim(Exc) ≥ 2) render intersection numbers of canonical divisor with curves undefined. (Canonical divisor is not $\mathbb{Q}$-Cartier.)
Conjecture:	"Flips" — codim-two surgeries; "repair" non-$\mathbb{Q}$-Cartier canonical divisors.
PROBLEM:	Existence and finite termination of flips?
Partial fix:	Existence up to dimension 5. Finite termination up to dimension 3.

PROBLEM:	Uniqueness of minimal models?
Partial fix:	Uniqueness fails. But, in dimension 3, birationally equivalent minimal varieties are related by known codim-two surgeries called "flops."
PROBLEM:	Higher-dim'l birationally equiv. minimal varieties still related by flops?
PROBLEM:	Existence, finite termination of flops? Number of birational models $< \infty$?

From minimal model to canonical model

Abundance Conjecture (from MMP)

If K_M is nef, then K_M is *semiample*, i.e. some pluricanonical map $\phi_{mK_M} : M \dashrightarrow \mathbb{P}^N$ is in fact holomorphic.

FACT: If M is of general type and K_M is nef, then K_M is semiample.
FACT: The Abundance Conjecture is true for 3-folds.

If $R(M) := \bigoplus_{m \geq 0} H^0(M, mK_M)$ is finitely generated, then the images of the pluricanonical maps ϕ_{mK_M}, for $m \gg 0$, are all isomorphic to:

$$\text{Proj}(R(M)) =: M_{\text{can}}.$$

Going from *minimal model* to *canonical model*:

- Gain: uniqueness for each birational class.
- Potential Gain: ampleness of $K_{M_{\text{can}}}$. (True for minimal X of general type.)
- Losses:
  - even worse singularities than the minimal models,
  - $0 \leq \dim(M_{\text{can}}) = \text{kod}(M) \leq \dim(M)$,
  - existence — requires finite generation of $R(M)$.

Bigness of Line Bundles

Loosely speaking, "bigness" of a line bundle is "birational ampleness."

DEFINITION: A line bundle L on a compact complex manifold M is said to be *big* if $\kappa(M, L) = \dim_{\mathbb{C}}(M)$.

- FACT: Bigness of L (and nonsingularity of M) $\Longrightarrow$ for some $m > 0$, the rational map $\phi_{mL} : M \dashrightarrow \mathbb{P}(H^0(M, mL))$ is surjective and birational.
- DEFINITION: A compact complex manifold M is said to be *of general type* if any one of the following equivalent conditions holds: kod$(M) = \dim_{\mathbb{C}}(M) \iff K_M$ is big $\iff$ for some large enough $m > 0$, the pluricanonical map $\phi_{mK_M} : M \dashrightarrow \mathbb{P}^N$ is a birational holomorphic map onto its image.
- The only thing you need to know (How many of you are in outer space by now?): For a compact complex manifold M, its canonical bundle K_M being "*nef and big*" means
  - M is minimal and of general type,
  - hence, its canonical model M_{can} exists, with $K_{M_{\text{can}}}$ ample,
  - M_{can} is a birational model of M,
  - M_{can} can be thought of as the image of ϕ_{mK_M}, for $m \gg 0$.

Canonical metrics, birational geometry & MMP

- FAMOUS FACT: A compact Kähler manifold M with $c_1(M) \leq 0$ ($K_M \geq 0$) admits a unique Kähler-Einstein metric.

  [Yau: continuity method], [Aubin, ?? method], [Cao: Kähler-Ricci flow].

- Recall: K_M nef and big $\Longrightarrow$ M_{can} exists, with $K_{M_{\text{can}}}$ ample.
- One might expect: The canonical model M_{can} of a projective manifold M may admit a (singular) Kähler-Einstein metric.

The (Fields-medal-winning) Calabi-Yau Theorem

On a compact Kähler manifold, the prescribed Ricci curvature problem has a unique solution in every Kähler class. *More precisely:*

Let M be a compact Kähler manifold. Then, given

- any form $\Omega \in -\mathbf{i}\,2\pi\, c_1(M)$,
- any Kähler class $[\omega] \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$,

there exists a unique Kähler form $\omega \in [\omega]$ such that $\text{Ric}(\omega) = \Omega$.

Recent Results

Canonical metrics on varieties of general type via Kähler-Ricci flow

[Tian-Zhang, '06]
If X is a minimal complex surface of general type, then the global solution of the Kähler-Ricci flow converges to a positive current $\widetilde{\omega}_\infty$ which descends to the Kähler-Einstein orbifold metric on its canonical model. In particular, $\widetilde{\omega}_\infty$ is smooth outside finitely many rational curves and has local continuous potential.

[Casini-La Nave, Mar '06]
Let M be a projective manifold of general type with canonical divisor K_M not nef. Then,

- $\exists$ some curve $C \subset M$ with $K_M \cdot C < 0$ and a holomorphic map $c : M \longrightarrow M'$ (M' possibly singular) which contracts C.
- the Kähler-Ricci flow $g(t)$ on M, with a certain initial metric depending on K_M and c, develops singularity in finite time, say T;
- the singular locus of $g(T)$ is contained in a proper subvariety of M;
- If M' is furthermore smooth, then $g(T)$ descends to a smooth metric on M'.

Recent Results

Canonical metrics on surfaces of intermediate Kodaira dimension via Kähler-Ricci flow

- A properly (kod$(S) = 1$) elliptic surface $S \xrightarrow{f} \Sigma$ is a fibration over a curve Σ with generic fiber being a smooth elliptic curve. It turns out that the base curve Σ is the canonical model of S.

- [Tian-Song, Mar '06] The Kähler-Ricci flow on a minimal properly elliptic surface $S \xrightarrow{f} S_{\text{can}}$ has global solution $\omega(t, \cdot)$ which converges as currents to $f^*\omega_\infty$, where ω_∞ is a positive current on S_{can}, smooth on the smooth locus of S_{can}, and is a "generalized" Kähler-Einstein metric in the following sense:

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP} + \left\{ \begin{array}{c} \text{further correction terms due to} \\ \text{presence of singular fibers of } f \end{array} \right\},$$

where ω_{WP} is the induced Weil-Petersson metric on S_{can}.

THE END

THANK YOU!