Andrea Young. Kähler-Ricci Flow. 05/01/07 ①

"Ricci Flow on Kähler Mflds".

$(M^n, J, g)$, ie $\dim_{\mathbb{C}} = n$, $J$ almost cx str, $g$ metric

$g_{\mathbb{C}}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) = g_{\alpha\bar{\beta}}$ Recall. $g_{\alpha\beta} = \overline{g_{\alpha\bar{\beta}}} = g_{\bar{\alpha}\beta}$ ; $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$

$\omega = \sqrt{-1}\, g_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$.

$Rc_{\mathbb{C}}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) = R_{\alpha\bar{\beta}}$ , $\rho = \frac{1}{2} Rc(JX, Y) = \sqrt{-1}\, R_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$

Note: $R_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log \det (g_{\alpha\bar{\beta}})$

Holomorphic sectional & Bisectional Curvature.

Let $Z \in T^{1,0}(M)\setminus\{0\}$, $X = Re(Z)$

The holo. sectional curv. $K_{\mathbb{C}}(Z) = \dfrac{Rm(X, JX, JX, X)}{|X|^4} = \dfrac{Rm_{\mathbb{C}}(Z, \bar{Z}, Z, \bar{Z})}{|Z|^4}$.

Then in unitary coords: $K_{\mathbb{C}} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$

- Pos (non-neg) Riemannian sectional curv
  $\Rightarrow$ pos (non-neg) hol. sectional curv.

Let $Z, W \in T^{1,0}M$ The Bisectional curvature

$K_{\mathbb{C}}(Z, W) = Rm_{\mathbb{C}}(Z, \bar{Z}, W, \bar{W}) / |Z|^2 |W|^2$.

Note: $\cdot\ K_\alpha(Z,Z) = K_\alpha(Z)$.

.. Pos (non-neg) bisect'l curv $\Rightarrow$ pos (non-neg) hol. sectional curv.

... Pos (non-neg) Riem sect curv $\Rightarrow$ pos (non neg) bisectional curv.

Recall (Frankel Conjecture)

- If $(M^n, g)$ is closed Kähler, pos bisectional curv, then $M^n$ is biholomorphic to $\mathbb{CP}^n$.

- Any $\mathbb{CP}^n$ admits a metric of constant holo sectional curvature (The Fubini-Study metrics)

$\partial\bar\partial$-lemma:

$\tau$ is exact, real $(1,1)$ form $\Rightarrow \exists\ \varphi$ st $\frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta}\varphi = \tau_{\alpha\bar\beta}$

and $\varphi$ is called a potential function

(KRF): $\frac{\partial}{\partial t} g_{\alpha\bar\beta} = -R_{\alpha\bar\beta}$.

Lemma: If $(M^n, J, g_0)$ is closed Kähler, then $\exists$ sol'n to KRF for short time.

Pf: KRF is equivalent to a scalar parabolic complex Monge-Ampère Eq'n.

KRF: $\frac{\partial}{\partial t}[\omega] = -[\rho(t)] = -[\rho(0)]$

$[\omega(t)] = [\omega(0)] - t[\rho(0)]$

$\omega(t) - \omega(0) + t\rho(0)$ is exact so by $\partial\bar{\partial}$-lemma

$$g_{\alpha\bar{\beta}}(t) = g^{0}_{\alpha\bar{\beta}} + t\,\partial_\alpha \partial_{\bar{\beta}} \log\det g_{\alpha\bar{\beta}} + \partial_\alpha \partial_{\bar{\beta}} \varphi$$

$$\Rightarrow \quad -R_{\alpha\bar{\beta}}(t) = \partial_\alpha \partial_{\bar{\beta}} \log\det\left(g^{0}_{\gamma\bar{\delta}} + t\,\partial_\gamma \partial_{\bar{\delta}}(\log\det g^{0}_{\mu\bar{\nu}}) + \partial_\gamma \partial_{\bar{\delta}} \varphi(t)\right)$$

$$\Rightarrow \frac{\partial \varphi}{\partial t} = \log\left(\frac{\det g^{0}_{\gamma\bar{\delta}} + t\,\partial_\gamma \partial_{\bar{\delta}} \log\det g^{0}_{\mu\bar{\nu}} + \partial_\gamma \partial_{\bar{\delta}}\varphi}{\det g^{0}_{\gamma\bar{\delta}}}\right) + C_1(t)$$

short-time existence follows

Fact: Kähler condition is preserved by KRF.

Normalized KRF: Assume $[\rho_0] = c[\omega_0]$ for $c \in \mathbb{R}$. $\rho/n$

Consider $g = g_1 * g_2$ on $S^2 \times S^2$. $H^{1,1}(S^2 \times S^2) = H^{1,1}(S^2) + H^{1,1}(S^2)$

$[\omega] = [\omega_1] + [\omega_2]$

$[\rho] = [\rho_1] + [\rho_2]$

on $S^2$, $[\omega]$ represents area, $[\rho]$ is a fixed elt of $2\pi[1]$.

$C([\omega_1] + [\omega_2]) = [\rho_1] + [\rho_2]$.

Hence: in order for NKRF to flow to the K-E metric on $S^2 \times S^2$, Both spheres had to have the same area to start with.

$$\frac{\partial A}{\partial t} = -\int R \, d\mu = -4\pi$$

NKRF: $\dfrac{\partial g_{\alpha\bar\beta}}{\partial t} = -R_{\alpha\bar\beta} + \dfrac{r}{n} g_{\alpha\bar\beta}$, $r = \int R\,d\mu / \mathrm{vol}$

Known Results: Existence & convergence:

Compact: $1^{st}$ Chern class has definite sign.

- Cao: NKRF long time existence.
- Cao: $C_1(M) \leq 0$: $g(t) \to g_\infty$ to ! Kähler Einstein metric in $[\omega_0]$
- Thm (Bando $n=3$, Mok $n \geq 4$) non-neg bisectional curvature then $g(t)$ soln to KRF has non-neg bisectional curv. $\forall t$.

Thm: (Tian, Chen) non-neg bisectional, pos at one point NKRF $\longrightarrow$ Fubini-Study.

Non Compact: Conjecture (Yau) M has pos bisectional curve $\Rightarrow$ biholomorphic to $\mathbb{C}^n$

Thm (Chau, Tian): max volume growth $\Rightarrow$ $M^n$ bihol $\mathbb{C}^n$.