Manifold man

Professor Robert Gompf is highly respected in the field of 4-manifold topology and is currently developing new techniques for understanding 4-manifolds – a rapidly progressing area of research

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To begin, what are manifolds and why are they important?

Manifolds are curves, surfaces and their generalisations to arbitrary dimensions. Whenever there is a problem involving many variables and constraints, a manifold is likely to be involved. Such systems occur commonly in diverse areas such as physics, engineering and economics.

For example, the minute hand on a clock has only one degree of freedom – it can be moved clockwise or counterclockwise, so its possible positions comprise a one-dimensional manifold (1-manifold or curve), in this case a circle. A robot arm with no elbow joint, moving in space from a fixed pivot, has two degrees of freedom, so its positions comprise a 2-manifold or surface, namely a sphere. A robot arm moving in a plane, but with an elbow joint, determines a different 2-manifold – its positions are described by two independent circles, which together comprise a torus, or inner-tube shape (one circle runs around the outer rim; the other follows a cross-section of the tube). A computer operating robot arms would face two very different problems for these two different kinds of arms, even though they both have two degrees of freedom. For a manifold with many dimensions, imagine a robot arm with several joints (if it moves through space, each joint contributes two dimensions).

Could you describe your latest research project and its chief aims and objectives?

In the past six months, I have mainly been writing a paper on exotic (non-standard) smooth structures on open 4-manifolds. The usual techniques for finding and distinguishing exotic structures on open 4-manifolds go back to the 1980s, but are very indirect and do not give much concrete information about the structures. Since minimal genera are very concrete, I have been studying what can be said about them for exotic smoothings of open 4-manifolds. There is virtually no prior work on this. In the longer term, I am planning to continue other recent research on open 4-manifolds and on Cappell-Shaneson homotopy 4-spheres.

What have been your main findings to date and what is their significance?

Perhaps my most dramatic recent finding concerns the 4-dimensional Poincaré Conjecture and its potential counterexamples, due to Cappell and Shaneson in the 1970s. In 2009, Selman Akbulut proved that infinitely many of these examples were standard, by adding a key observation to a long, difficult computation from the 1990s, representing several years of work by Akbulut and Robion Kirby, followed by some of my own. When I saw Akbulut's new paper I immediately recognised the key step as similar to steps in my own work, and found a deeper explanation. I soon proved that a much larger collection was standard, by a far more efficient method that started from the original definition. I suspect that the method actually proves that all Cappell-Shaneson spheres are standard, although the algebra becomes very difficult in some cases. In fact, I am now leaning toward the opinion that the four-dimensional Poincaré Conjecture is true, since all known methods of constructing homotopy 4-spheres are known to produce only the standard 4-sphere under various conditions. However, I have also been exploring a generalisation of the Cappell-Shaneson method that produces a much larger collection of homotopy spheres, many of which do not seem amenable to simplification, so the game continues.

Do you hope to continue on this trajectory or will you begin investigating new quandaries after completing this project?

I am already thinking about various other problems related to the classification of 4-manifolds, and about the interaction of 4-manifold topology with other areas of mathematics. Unlike in so-called 'Big Science' with its huge research teams, I do not have the inertia of a big lab to slow me down, and I can go in whatever directions seem most productive at the moment.

What are the wider applications of your work beyond the scope of pure mathematics?

Since manifolds in general underlie problems in many other disciplines, there is a fertile field from which applications should grow. For a more immediate example, general relativity models the universe (space and time together) as a 4-manifold. It would be interesting to know which 4-manifold we actually live in. I recently had an email exchange with a relativist who is interested in exotic 4-manifolds. She is researching the hypothesis that this phenomenon could explain the existence of mass in the Universe. My wife sometimes tells people that if time travel is discovered, my research will be at the foundation! That may sound far-fetched, but who knows?

Manifold **intelligence**

A research project at the **University of Texas at Austin** is at the forefront of a study of the complex field of 4-manifold topology, aiming to address questions that have remained unanswered for over 30 years

IN MATHEMATICS, an n-dimensional manifold is a topological space. Lines and circles are 1-dimensional manifolds; surfaces such as the plane, the sphere and the torus are 2-manifolds; a tumbling asteroid, which can rotate about three different axes determines a 3-manifold; and so forth. Near each point, a manifold resembles Euclidean space – the properties of which are relatively well understood – and the concept of a manifold is therefore central to many parts of geometry and mathematical physics because it allows complicated structures to be described and understood.

CLASSIFYING MANIFOLDS

The most fundamental problem about manifold topology is that of classification: what intrinsically different kinds of manifolds are there? At the most basic level, size is not important, but the difference between a sphere and a torus is crucial. Intuitively, a torus is different because it has a 'hole' in it, but even this statement is difficult to formulate mathematically. In order to study manifolds by performing calculus, researchers work with smooth manifolds, so-called because they do not have corners or crinkles. In contrast, topological manifolds have corners and crinkles of various sorts.

For manifolds of dimension 2 or less (curves and surfaces), the classification problem was completely solved in the 19th Century. For example, every 'closed' curve is essentially the same as a circle, but the sphere and torus are in essence different from each other and from surfaces with more 'holes' in them. The study of 3-manifolds progressed throughout the 20th Century, and while there is not yet a complete classification, there is a highly developed theory regarding the diversity of 3-manifolds.

During the 1960s, mathematicians found that in dimensions 5 and above, there is enough room to prove several powerful theorems that reduce many topological questions to algebraic problems that can frequently be solved. In the remaining dimension 4, which is too high-dimensional for low-dimensional techniques to be useful, but too confined for the high-dimensional theory to apply, virtually no progress occurred until 1981, when two breakthroughs made by Michael Freedman and Simon Donaldson revolutionised the field. However, after three decades of the application of the techniques they developed, the main conclusion appears to be that 4-manifold theory is much more complicated than anyone had imagined.

HOMOTOPY 4-SPHERES

Professor Robert Gompf has witnessed the field of 4-manifold theory unfold. As a graduate student when the work of Freedman and Donaldson was published, he knew just enough at the time to realise that it was revolutionary, and since then has devoted much of his career to understanding 4-manifold topology. Now highly respected in the field, Gompf is currently researching homotopy 4-spheres.

Homotopy spheres are manifolds that cannot be distinguished from the standard sphere of the same dimension by the basic techniques of algebraic topology. At the beginning of the 20th Century, Henri Poincaré conjectured that every homotopy 3-sphere is actually a 3-sphere – if you cannot tell that it is not a 3-sphere by algebraic techniques, then it is really a 3-sphere. Exotic spheres are homotopy spheres that are not standard. In dimension 4, there are several ways of constructing infinite collections of homotopy 4-spheres that are not known to be standard, and all of the current methods of distinguishing 4-manifolds are known to fail for homotopy spheres. The plethora of potential counterexamples has led to the prevailing view that Poincaré's conjecture is probably false in dimension 4, but Gompf's recent research suggests otherwise.

The most promising collection of homotopy 4-spheres was constructed by Sylvain Cappell and Julius Shaneson in the 1970s. While other constructions were known to produce standard spheres under some conditions, the Cappell-Shaneson examples have been more difficult to understand. The examples were

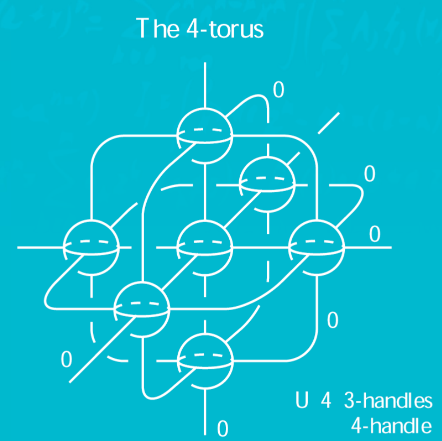

FIGURE 2. 4-torus ©American Mathematical Society, Robert E Gompf and András I Stipsicz, 4-Manifolds and Kirby Calculus, 1999, (*Graduate Studies in Mathematics*, volume 20, page 137, figure 4.42).

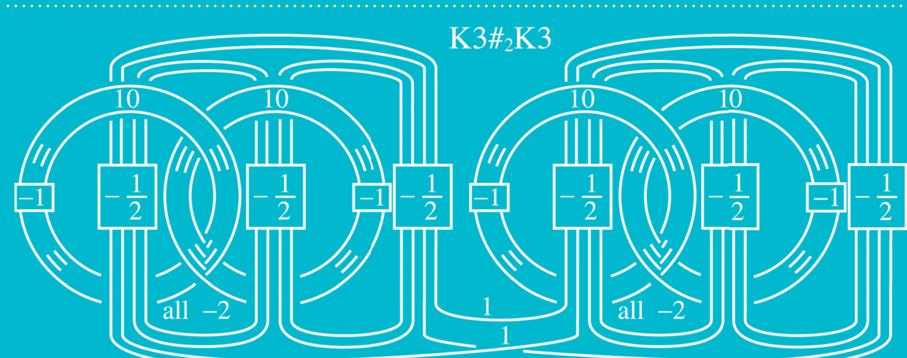

FIGURE 1. K3#2K3 ©American Mathematical Society, Robert E Gompf and András I Stipsicz, 4-Manifolds and Kirby Calculus, 1999, (*Graduate Studies in Mathematics*, volume 20, page 411, figure 10.4).

INTELLIGENCE

4-MANIFOLD TOPOLOGY AND RELATED TOPICS

OBJECTIVES

To improve understanding of four-dimensional manifolds and their smoothings, and of geometric structures on smooth 4-manifolds such as symplectic and Stein structures. Short-term objectives include showing that various homotopy 4-spheres are standard and determining whether all open 4-manifolds admit exotic smooth structures. To understand the variety of smoothings on a given open 4-manifold, Gompf is investigating the range of values of the minimal genus function for a given homology class.

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ROBERT GOMPF completed a BSc at the Massachusetts Institute of Technology in 1979 and a PhD at the University of California at Berkeley in 1984. Since then he has held several posts at the University of Texas at Austin and is currently a Jane and Roland Blumberg Centennial Professor. He has also been a Visiting Professor at the Mathematical Sciences Research Institute, Berkeley, and Max-Planck-Institute for Mathematics, Germany. Prior to that Gompf was an Association Professor at the University of Texas at Austin. His current research interests focus on 4-manifold, symplectic and contact topology.

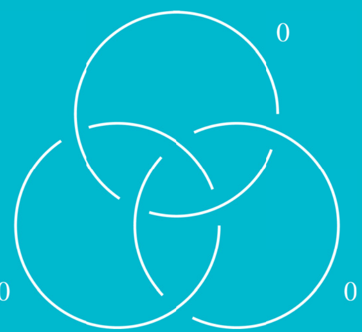

FIGURE 3. 3-torus ©American Mathematical Society, Robert E Gompf and András I Stipsicz, 4-Manifolds and Kirby Calculus, 1999, (*Graduate Studies in Mathematics*, volume 20, page 159, figure 5.25).

extensively studied by various researchers including Gompf in the 1980s and 1990s, though little progress was made at that time. It is these examples that Gompf is now revisiting, with support from the National Science Foundation, and he plans to further extend a method that he previously developed, attempting to show that all Cappell-Shaneson spheres are standard.

EXOTIC SMOOTH STRUCTURES

Another focus of Gompf's current research is exotic smooth structures on open 4-manifolds. Closed manifolds, like the n-sphere and torus, have no loose ends and are therefore finite; the remaining manifolds are called 'open'. The main techniques for distinguishing different smooth structures on a closed 4-manifold are achieved through the classification of surfaces. Closed, 'orientable' surfaces are classified by their genus, which is the number of 'holes' in them. Higher dimensional analogs of holes are detected by homology. As Gompf explains: "Given a 2-homology class on a 4-manifold, we can ask what the smallest possible genus is for a smoothly embedded surface representing that homology class. The answer measures which smooth structure we have".

The topological Poincaré Conjecture, which is now known to be true, says that every homotopy n-sphere is topologically equivalent to the standard n-sphere. However, in dimensions 7 and up, and possibly in dimension 4, there are smooth manifolds topologically, but not smoothly equivalent to the n-sphere. That is, they can be made to look like the n-sphere, but only by crinkling them. Alternatively, there are essentially different ways of performing calculus on the topological n-sphere. These are called exotic smooth structures. Open 4-manifolds tend to have infinitely many different exotic smooth structures – in fact, as many as there are real numbers, whereas closed 4-manifolds only have as many as the integers. The most striking example is that Euclidean 4-space has as many smooth structures as there are real numbers, whereas in other dimensions Euclidean space only has one smooth structure.

4-MANIFOLD ANSWERS

Gompf is finding that he can construct exotic smoothings with a lot of control of minimal genera of homology classes – much more control than is possible for closed 4-manifolds. "It seems like just about anything that is not impossible for basic reasons actually occurs," he states. "I have constructed many examples exhibiting an open 4-manifold with infinitely many smoothings that can be distinguished by their minimal genera, which can be chosen with great flexibility. That is, I can specify in advance how I want the minimal genera to behave, then find an exotic smooth structure realising the specifications."

He is also combining the new techniques he has developed with his older ones to arrange that each choice of the minimal genera is realised by as many smoothings as there are real numbers.

Gompf's research is bringing the field much closer to an affirmative answer to the three decade-old question of whether all open 4-manifolds admit exotic smooth structures. He believes that as a boundary between low- and high-dimensional topology, 4-manifold theory displays many unique and surprising phenomena. "On the bright side, there will continue to be opportunities for fascinating research in the subject for many years," he adds.

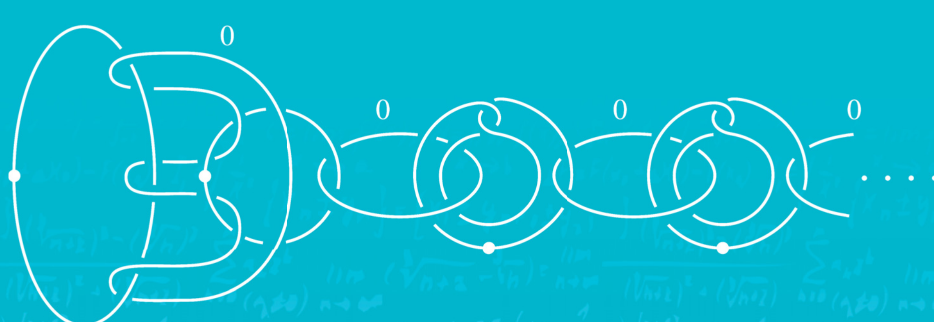

FIGURE 4. Exotic Euclidean 4-space ©American Mathematical Society, Robert E Gompf and András I Stipsicz, 4-Manifolds and Kirby Calculus, 1999, (*Graduate Studies in Mathematics*, volume 20, page 207, figure 6.16).