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Multi-Multiplier Ambient-Space Formulations of Constrained Dynamical Systems, with an Application to Elastodynamics

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Abstract

Various formulations of the equations of motion for both finite- and infinite-dimensional constrained Lagrangian dynamical systems are studied. The different formulations correspond to different ways of enforcing constraints through multiplier fields. All the formulations considered are posed on ambient spaces whose members are unrestricted by the need to satisfy constraint equations, but each formulation is shown to possess an invariant set on which the constraint equations and physical balance laws are satisfied. The stability properties of the invariant set within its ambient space are shown to be different in each case. We use the specific model problem of linearized incompressible elastodynamics to compare properties of three different ambient-space formulations. We establish the well-posedness of one formulation in the particular case of a homogeneous, isotropic body subject to specified tractions on its boundary.

1. Introduction

The equations of motion for a Lagrangian system on a manifold defined by configuration (or holonomic) constraints can be formulated in various different ways. The three main possibilities are:

(a) Euler-Lagrange equations in local coordinates;
(b) Euler-Lagrange equations in ambient coordinates with explicit multipliers to enforce the constraints (such equations may be obtained from Hamilton's variational principle with a multiplier rule);
(c) lifted or extended equations in ambient coordinates with an invariant manifold (such equations may be obtained by eliminating the multipliers in (b), and the physically meaningful solutions reside in the invariant set).

A formulation as in (a) is often unavailable or inconvenient for infinite- or large finite-dimensional systems, and in these cases a formulation as in (b) may be more natural. However, dealing with explicit constraints and multipliers can sometimes be cumbersome in both analytical and numerical investigations, and this has led many authors to consider formulations as in (c). We refer to (c) as *ambient-space formulations*. Such formulations have appeared in various guises [4, 6, 11, 16, 20, 23, 28, 31, 33, 35], but apparently there has been no systematic comparison between their properties.

In this article we develop a number of results on ambient-space formulations for Lagrangian systems subject to configuration constraints. We show that, dependent on the precise manner in which constraints are introduced into Hamilton's principle, ambient-space formulations with markedly different structural properties can be constructed for a given constrained system. In particular, both Hamiltonian and non-Hamiltonian ambient-space formulations can be constructed, and each can have either a stable or unstable invariant set. The different formulations correspond to different ways in which single multipliers of either pressure or striction type, or multiple multipliers of differing types, can be employed in Hamilton's principle. Here we describe a multiplier as pressure-like if it is associated with a configuration-level constraint, and as striction-like if it is associated with a velocity-level constraint [31, 11].

An important structural property of an ambient-space formulation is the stability of the associated invariant set. We show that stability depends crucially on the type and number of multipliers used in constructing the formulation. In the context of infinite-dimensional systems, the stability issue expands to include problems of solvability and well-posedness, especially for initial data off the invariant set. In contrast to the situation arising in finite-dimensional dynamics, well-posedness need not be automatic for an ambient-space formulation in infinite dimensions. For example, the velocity-impulse formulation of incompressible fluid dynamics given by OSELEDETS [35] has been shown by E & LIU [13] to be marginally ill posed.

Our motivations for studying ambient-space formulations are twofold. First, these types of formulations can be useful within the context of numerical analysis. For example, such formulations can assist in the analysis of discretization schemes for a corresponding differential-algebraic formulation [29, 20, 3], or they can themselves be discretized to provide a basis for simulation [28, 4, 13]. For the analysis and design of numerical schemes, we believe it is important to understand the stability properties of the invariant set. Second, ambient-space formulations can be useful in analytical investigations. For example, ambient-space formulations can be exploited to study dynamic stability [11], to develop regularity estimates [16, 23] and to characterize integrability [33] in constrained systems.

The presentation is structured as follows. In Section 2 we introduce and discuss ambient-space formulations for finite-dimensional mechanical systems. Our main definition in this section is motivated by various works that have appeared previously; for example, the work of Kozlov summarized in [2], works based on the theory of Dirac [40, 28], work based on the stabilization of differential-algebraic equations [4], and the impetus-striction method developed in [31, 11, 12]. Within the context of a simple model problem we illustrate three different approaches to the construction of ambient-space formulations. These approaches lead to formulations for which the invariant sets exhibit markedly different stability properties. In particular, formulations can be constructed for which the invariant set is unstable, the level set of a first integral, or exponentially attractive in the sense that the values of the constraints approach their values on the invariant set at an exponential rate. Indeed, we describe new formulations that are Hamiltonian in the whole of the ambient space while possessing exponentially attractive invariant sets.

In the remainder of the article we extend the finite-dimensional analysis to a particular infinite-dimensional problem in linearized incompressible elastodynamics. In Section 3 we present a standard formulation of the problem along with an underlying variational structure that will be used to motivate three different types of ambient-space formulations. The first type involves a pure pressure field, the second a pure striction field, and the third employs a combination of pressure and striction fields. The pure pressure formulation has an invariant set that is unstable in an appropriate sense, while the pure striction formulation has an invariant set that is the level set of a (pointwise) integral. Moreover, by including an appropriate arbitrary parameter in the pure striction formulation, the invariant set can be made exponentially attractive as in the case of finite dimensions.

For the elastodynamics problem, the pure striction approach leads to a formulation with some peculiarities: boundary values for the striction variable are determined by an explicit evolution equation on the boundary. This boundary equation appears to be inconvenient and it is unclear whether the formulation is well posed. To circumvent this difficulty we develop a third formulation that employs both pressure and striction multiplier fields. The use of multiple multipliers in this case provides a means to avoid the peculiarities of the pure striction approach without sacrificing stability of the invariant set. Moreover, we gain well-posedness, as we prove in Sections 4 and 5 for the initial-boundary value problem corresponding to a homogeneous, isotropic body with a specified traction on its boundary. This well-posedness result provides a concrete example that illustrates the utility of ambient-space formulations in analytical investigations.

The variables of displacement, momentum and pressure employed in pressure-type formulations of constrained problems are all familiar physical quantities. In contrast, the variables of displacement, impetus and striction employed in striction-type formulations are not well known. A physical interpretation of these new variables is presented in Section 6.

2. Finite-Dimensional Systems

In this section we introduce the notion of an ambient-space formulation within the context of a finite-dimensional model problem. Three ambient-space formulations are constructed and their properties are summarized in a sequence of propositions. The proofs of the results stated in this section are straightforward and are omitted for brevity. While the results presented here are for the case of a single constraint and a simple, decoupled Lagrangian function of the form kinetic minus potential energies, all the results extend to multiple constraints and general Lagrangians that are convex in the velocities.

2.1. Model Problem: Constrained Formulation

Consider a particle of mass m and position $\boldsymbol{q}(t) \in \mathbb{R}^n$ that is constrained to lie on a smooth frictionless surface $Q \subset \mathbb{R}^n$. We suppose that Q is defined by the zero level set of a function $g : \mathbb{R}^n \to \mathbb{R}$ and that the particle moves under the action of a conservative force field with potential $V : \mathbb{R}^n \to \mathbb{R}$. For such a system, Hamilton's principle provides a variational characterization of the motion in terms of the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \tfrac{1}{2}\dot{\boldsymbol{q}} \cdot m\dot{\boldsymbol{q}} - V(\boldsymbol{q}), \tag{2.1}$$

and there are two basic ways to formulate equations of motion. One way is to introduce a local coordinate system in Q, restrict the Lagrangian (2.1) to the tangent bundle TQ, and invoke Hamilton's principle. This approach leads to Euler-Lagrange equations in local coordinates. However, such coordinates are often inconvenient to construct for large finite-dimensional systems, and may not be available at all for infinite-dimensional systems.

A second approach is to work in the coordinates of the ambient space $\mathbb{R}^n \times \mathbb{R}^n$. Here we consider Hamilton's principle for the Lagrangian (2.1) and enforce the constraint $g(\boldsymbol{q}) = 0$ with a multiplier λ. This approach leads to the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt} D_2\hat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda) &= D_1\hat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda), \\ 0 &= D_3\hat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda), \end{aligned} \tag{2.2}$$

where $\hat{L} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the augmented Lagrangian function

$$\hat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \lambda g(\boldsymbol{q}). \tag{2.3}$$

When we introduce the momentum variable $\boldsymbol{p} = D_2\hat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda) \in \mathbb{R}^n$, the system in (2.2) may be written as a first-order system in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, namely

$$\begin{aligned} \dot{\boldsymbol{q}} &= \quad m^{-1}\boldsymbol{p}, \\ \dot{\boldsymbol{p}} &= -DV(\boldsymbol{q}) - \lambda Dg(\boldsymbol{q}), \\ 0 &= \quad g(\boldsymbol{q}), \end{aligned} \tag{2.4}$$

where $Dg(\boldsymbol{q})$ is the derivative of the constraint function g at $\boldsymbol{q}$, and $\lambda(t) \in \mathbb{R}$ is interpreted as a multiplier that is determined by the condition $(2.4)_3$. This formulation is referred to as *constrained* since there is no explicit evolution equation for λ. Alternatively, due to the presence of the algebraic relation in $(2.4)_3$, such formulations are often described as *differential-algebraic*. Notice that smooth solutions of (2.4) are restricted to lie in the set

$$\mathcal{A}_0 = \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^n \times \mathbb{R}^n \mid f(\boldsymbol{q}, \boldsymbol{p}) = 0 \quad \text{and} \quad g(\boldsymbol{q}) = 0\}, \tag{2.5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\boldsymbol{q}, \boldsymbol{p}) = Dg(\boldsymbol{q}) \cdot m^{-1}\boldsymbol{p}$.

For details on the theory of differential-algebraic systems such as (2.4) see [36], and for details on the special difficulties associated with the numerical treatment of these systems see [6]. Motivated by the difficulties in treating systems of the form (2.4), we pursue alternate formulations.

2.2. Ambient-Space Formulations: Definition

Consider a system of ordinary differential equations in $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}\boldsymbol{p}(\boldsymbol{q},\boldsymbol{\xi}), \\ \dot{\boldsymbol{\xi}} &= \boldsymbol{w}(\boldsymbol{q},\boldsymbol{\xi}), \end{aligned} \tag{2.6}$$

where $\boldsymbol{p}, \boldsymbol{w} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are smooth in an open set $\mathcal{N}$. We say that (2.6) is an ambient-space formulation for the constrained system (2.4) if the following conditions hold:

1. The system possesses an invariant set $\mathcal{M}_0 \subset \mathcal{N}$.
2. The map $\boldsymbol{\chi} : \mathcal{M}_0 \to \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\boldsymbol{\chi}(\boldsymbol{q},\boldsymbol{\xi}) = (\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{q},\boldsymbol{\xi}))$$

   is onto the set $\mathcal{A}_0$ defined in (2.5).
3. If $(\boldsymbol{q},\boldsymbol{\xi})(t)$ is a solution in $\mathcal{M}_0$, then $(\boldsymbol{q},\boldsymbol{p})(t) = \boldsymbol{\chi}(\boldsymbol{q},\boldsymbol{\xi})(t)$ is a solution of (2.4) for some $\lambda(t)$.

The invariant set $\mathcal{M}_0$ will be called the physical solution set of the ambient-space formulation.

2.3. Ambient-Space Formulations: Examples

In preparation for the infinite-dimensional system to be considered later in this article, we next construct three ambient-space formulations for the finite-dimensional system (2.4). For a general constraint $g(\boldsymbol{q})$ the formulations can be quite complicated; however, they simplify considerably when $g(\boldsymbol{q})$ is linear, which is a case of interest in infinite dimensions.

2.3.1. Standard Example. Our first ambient-space formulation follows directly from (2.4) by eliminating the multiplier λ. To begin, we differentiate the constraint equation $(2.4)_3$ twice with respect to time, substitute from $(2.4)_{1,2}$ and solve for λ as a function of the phase variables to get $\lambda = \hat{\lambda}(\boldsymbol{q},\boldsymbol{p})$, where

$$\begin{aligned} \hat{\lambda}(\boldsymbol{q},\boldsymbol{p}) = [Dg(\boldsymbol{q}) \cdot m^{-1} Dg(\boldsymbol{q})]^{-1} \big[& m^{-1}\boldsymbol{p} \cdot D^2 g(\boldsymbol{q})(m^{-1}\boldsymbol{p}) \\ & - Dg(\boldsymbol{q}) \cdot m^{-1} DV(\boldsymbol{q}) \big] \end{aligned} \tag{2.7}$$

and $D^2 g(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the Hessian of $g(\boldsymbol{q})$ at $\boldsymbol{q}$. Given this expression for λ, we consider the system of ordinary differential equations given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= \quad m^{-1}\boldsymbol{\xi}, \\ \dot{\boldsymbol{\xi}} &= -DV(\boldsymbol{q}) - \hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi}) Dg(\boldsymbol{q}). \end{aligned} \tag{2.8}$$

The relation between (2.8) and (2.4) is contained in the following result.

Proposition 2.1. *Let* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be defined by* $f(\boldsymbol{q}, \boldsymbol{\xi}) = Dg(\boldsymbol{q}) \cdot m^{-1}\boldsymbol{\xi}$ *and for any regular value* c *of* g *let*

$$\mathcal{M}_c = \{(\boldsymbol{q}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \mid f(\boldsymbol{q}, \boldsymbol{\xi}) = 0 \quad \textit{and} \quad g(\boldsymbol{q}) = c\}.$$

Then (2.8) *has the following properties:*

1. *The function* $f(\boldsymbol{q}, \boldsymbol{\xi})$ *is an integral for* (2.8).
2. *The set* $\mathcal{M}_c$ *is an invariant set for* (2.8).
3. *The map* $\boldsymbol{\chi} : \mathcal{M}_0 \to \mathbb{R}^n \times \mathbb{R}^n$ *defined by*

$$\boldsymbol{\chi}(\boldsymbol{q}, \boldsymbol{\xi}) = (\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{\xi})) \quad \textit{where} \quad \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{\xi}) = \boldsymbol{\xi}$$

   is onto $\mathcal{A}_0$.
4. *If* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *is a solution of* (2.8) *in* $\mathcal{M}_0$, *then* $(\boldsymbol{q}, \boldsymbol{p})(t) = \boldsymbol{\chi}(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *is a solution of* (2.4) *with* $\lambda(t)$ *given by*

$$\lambda(t) = \hat{\lambda}(\boldsymbol{q}(t), \boldsymbol{\xi}(t)).$$

The above result shows that (2.8) is an ambient-space formulation for (2.4). Notice that (2.8) is in general not Hamiltonian away from the physical solution set $\mathcal{M}_0$. Moreover, this invariant set is unstable in an appropriate sense as shown in the following result.

Proposition 2.2. *Let* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *be a solution of* (2.8) *with corresponding initial data* $(\boldsymbol{q}_0, \boldsymbol{\xi}_0)$ *in a neighborhood of* $\mathcal{M}_0$. *If the initial data satisfy*

$$g(\boldsymbol{q}_0) = a \quad \textit{and} \quad f(\boldsymbol{q}_0, \boldsymbol{\xi}_0) = b$$

for some constants $a, b \in \mathbb{R}$, *then the solution* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *has the property that*

$$g(\boldsymbol{q}(t)) = a + bt \quad \textit{and} \quad f(\boldsymbol{q}(t), \boldsymbol{\xi}(t)) = b.$$

Thus, solutions of (2.8) with initial data arbitrarily close to the physical solution set $\mathcal{M}_0$ may move arbitrarily far away depending on the constraint function $g(\boldsymbol{q})$.

2.3.2. Striction-Based Example. Our second ambient-space formulation for (2.4) follows from a generalization of the impetus-striction method developed in [31, 11]. First, rather than consider the given constraint $g(\boldsymbol{q}) = 0$, we consider the related constraint $h(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$ where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(\boldsymbol{q}, \boldsymbol{v}) = Dg(\boldsymbol{q}) \cdot \boldsymbol{v} + \alpha g(\boldsymbol{q}) \tag{2.9}$$

and $\alpha \in \mathbb{R}$ is a parameter. Next, we consider Hamilton's principle for the Lagrangian (2.1) and enforce the constraint $h(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$ with a multiplier μ. This approach leads to the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt} D_2\tilde{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mu) &= D_1\tilde{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mu), \\ 0 &= D_3\tilde{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mu), \end{aligned} \tag{2.10}$$

where $\tilde{L} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the augmented Lagrangian function

$$\tilde{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mu) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mu h(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2.11}$$

Introducing the conjugate variable $\boldsymbol{\xi} = D_2\tilde{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mu) \in \mathbb{R}^n$, the system in (2.10) may be written as a first-order system in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, namely

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}[\boldsymbol{\xi} + \mu Dg(\boldsymbol{q})], \\ \dot{\boldsymbol{\xi}} &= -DV(\boldsymbol{q}) - \alpha\mu Dg(\boldsymbol{q}) - \mu D^2 g(\boldsymbol{q})[m^{-1}(\boldsymbol{\xi} + \mu Dg(\boldsymbol{q}))], \\ 0 &= Dg(\boldsymbol{q}) \cdot m^{-1}[\boldsymbol{\xi} + \mu Dg(\boldsymbol{q})] + \alpha g(\boldsymbol{q}). \end{aligned} \tag{2.12}$$

Following [31, 11], we refer to $\boldsymbol{\xi}$ as the impetus and μ as the striction. Using $(2.12)_3$ we can solve for μ as a function of the phase variables $\boldsymbol{q}$ and $\boldsymbol{\xi}$ to get

$$\mu(\boldsymbol{q}, \boldsymbol{\xi}) = -[Dg(\boldsymbol{q}) \cdot m^{-1} Dg(\boldsymbol{q})]^{-1}[Dg(\boldsymbol{q}) \cdot m^{-1}\boldsymbol{\xi} + \alpha g(\boldsymbol{q})]. \tag{2.13}$$

Given this expression for μ, we consider the system of ordinary differential equations given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}[\boldsymbol{\xi} + \mu(\boldsymbol{q}, \boldsymbol{\xi}) Dg(\boldsymbol{q})], \\ \dot{\boldsymbol{\xi}} &= -DV(\boldsymbol{q}) - \alpha\mu(\boldsymbol{q}, \boldsymbol{\xi}) Dg(\boldsymbol{q}) \\ &\quad - \mu(\boldsymbol{q}, \boldsymbol{\xi}) D^2 g(\boldsymbol{q})[m^{-1}(\boldsymbol{\xi} + \mu(\boldsymbol{q}, \boldsymbol{\xi}) Dg(\boldsymbol{q}))]. \end{aligned} \tag{2.14}$$

Properties of the formulation (2.14) and its relation to the constrained formulation (2.4) are summarized in the following result.

Proposition 2.3. *Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Hamiltonian function for the free particle, that is,*

$$H(\boldsymbol{q}, \boldsymbol{p}) = \tfrac{1}{2}\boldsymbol{p} \cdot m^{-1}\boldsymbol{p} + V(\boldsymbol{q}),$$

and for any regular value c of g let

$$\mathcal{M}_c = \{(\boldsymbol{q}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \mid g(\boldsymbol{q}) = c\}.$$

Then (2.14) *has the following properties:*

1. *The system is Hamiltonian with Hamiltonian function $\mathcal{H}(\boldsymbol{q}, \boldsymbol{\xi})$ defined by*

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{\xi}) = \min_{\mu}[\, H(\boldsymbol{q}, \boldsymbol{\xi} + \mu Dg(\boldsymbol{q})) + \alpha\mu g(\boldsymbol{q})\,].$$

2. *The function $g(\boldsymbol{q})$ is an integral for* (2.14) *if $\alpha = 0$.*
3. *The set $\mathcal{M}_0$ is an invariant set for* (2.14) *for any $\alpha \in \mathbb{R}$.*
4. *The map $\boldsymbol{\chi} : \mathcal{M}_0 \to \mathbb{R}^n \times \mathbb{R}^n$ defined by*

$$\boldsymbol{\chi}(\boldsymbol{q}, \boldsymbol{\xi}) = (\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{\xi})) \quad \textit{where} \quad \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{\xi}) = \boldsymbol{\xi} + \mu(\boldsymbol{q}, \boldsymbol{\xi}) Dg(\boldsymbol{q})$$

is onto $\mathcal{A}_0$.

5. *If* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *is a solution of* (2.14) *in* $\mathcal{M}_0$*, then* $(\boldsymbol{q}, \boldsymbol{p})(t) = \boldsymbol{\chi}(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *is a solution of* (2.4) *with* $\lambda(t)$ *given by*

$$\lambda(t) = \alpha\mu(\boldsymbol{q}(t), \boldsymbol{\xi}(t)) - \frac{d}{dt}\mu(\boldsymbol{q}(t), \boldsymbol{\xi}(t)).$$

The above result shows that (2.14) is an ambient-space formulation for (2.4). Notice that the Hamiltonian $\mathcal{H}(\boldsymbol{q}, \boldsymbol{\xi})$ is in general not decoupled or separable even though the original Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is, and that the physical solution set $\mathcal{M}_0$ for (2.14) is different from the one for (2.8). The next result shows that the invariant set $\mathcal{M}_0$ for (2.14) is exponentially attractive in an appropriate sense if $\alpha > 0$.

Proposition 2.4. *Let* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *be a solution of* (2.14) *with corresponding initial data* $(\boldsymbol{q}_0, \boldsymbol{\xi}_0)$ *in a neighborhood of* $\mathcal{M}_0$*. If the initial data satisfy* $g(\boldsymbol{q}_0) = a$ *for some* $a \in \mathbb{R}$*, then the solution* $(\boldsymbol{q}, \boldsymbol{\xi})(t)$ *has the property that* $g(\boldsymbol{q}(t)) = ae^{-\alpha t}$.

2.3.3. Multi-Multiplier Example. Our third formulation combines aspects of both the previous two approaches. This generalization is perhaps excessive in the finite-dimensional case, but in extending the previous striction-based formulation to infinite-dimensional problems such as linearized incompressible elasticity some peculiarities arise in the boundary conditions. The ambient-space formulation developed in this section, based on using two multipliers in Hamilton's principle, will allow us to avoid these peculiarities.

The ambient-space formulation of this section may be motivated as follows. First, notice that if the motion of a Lagrangian system satisfies the constraint $g(\boldsymbol{q}) = 0$, then it also satisfies the related constraint $h(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$ where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(\boldsymbol{q}, \boldsymbol{v}) = Dg(\boldsymbol{q}) \cdot \boldsymbol{v}.$$

With this in mind, we introduce into the variational principle of Hamilton a multiplier λ for the constraint $g(\boldsymbol{q}) = 0$, and a second multiplier μ for the constraint $h(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$. This approach leads to the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt} D_2\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu) &= D_1\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu), \\ 0 &= D_3\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu), \\ 0 &= D_4\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu), \end{aligned} \tag{2.15}$$

where $\bar{L} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the augmented Lagrangian function

$$\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \lambda g(\boldsymbol{q}) - \mu h(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2.16}$$

Introducing the conjugate variable $\boldsymbol{\xi} = D_2\bar{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda, \mu) \in \mathbb{R}^n$, the system in (2.15) may be written as a first-order system, namely

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}[\boldsymbol{\xi} + \mu Dg(\boldsymbol{q})], \\ \dot{\boldsymbol{\xi}} &= -DV(\boldsymbol{q}) - \lambda Dg(\boldsymbol{q}) - \mu D^2 g(\boldsymbol{q})[m^{-1}(\boldsymbol{\xi} + \mu Dg(\boldsymbol{q}))], \\ 0 &= g(\boldsymbol{q}), \\ 0 &= Dg(\boldsymbol{q}) \cdot m^{-1}[\boldsymbol{\xi} + \mu Dg(\boldsymbol{q})]. \end{aligned} \tag{2.17}$$

Using $(2.17)_4$ we can solve for μ as a function of the phase variables $\boldsymbol{q}$ and $\boldsymbol{\xi}$ to get

$$\mu(\boldsymbol{q},\boldsymbol{\xi}) = -[Dg(\boldsymbol{q})\cdot m^{-1}Dg(\boldsymbol{q})]^{-1}Dg(\boldsymbol{q})\cdot m^{-1}\boldsymbol{\xi}. \tag{2.18}$$

Given this expression for μ, and letting $\lambda = \hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi})$ for an arbitrary smooth function $\hat{\lambda}$, we consider the system of ordinary differential equations given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}[\boldsymbol{\xi} + \mu(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q})], \\ \dot{\boldsymbol{\xi}} &= -DV(\boldsymbol{q}) - \hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q}) \\ &\quad - \mu(\boldsymbol{q},\boldsymbol{\xi})D^2g(\boldsymbol{q})[m^{-1}(\boldsymbol{\xi} + \mu(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q}))]. \end{aligned} \tag{2.19}$$

Properties of the formulation (2.19) and its relation to the constrained formulation (2.4) are summarized in the following result.

Proposition 2.5. *Let $\hat{\lambda} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an arbitrary, smooth function and for any regular value c of g let*

$$\mathcal{M}_c = \{(\boldsymbol{q},\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \mid g(\boldsymbol{q}) = c\}.$$

Then (2.19) *has the following properties:*

1. *The function $g(\boldsymbol{q})$ is an integral for* (2.19).
2. *The set $\mathcal{M}_0$ is an invariant set for* (2.19).
3. *The map $\boldsymbol{\chi} : \mathcal{M}_0 \to \mathbb{R}^n \times \mathbb{R}^n$ defined by*

$$\boldsymbol{\chi}(\boldsymbol{q},\boldsymbol{\xi}) = (\boldsymbol{q},\boldsymbol{p}(\boldsymbol{q},\boldsymbol{\xi})) \quad \textit{where} \quad \boldsymbol{p}(\boldsymbol{q},\boldsymbol{\xi}) = \boldsymbol{\xi} + \mu(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q})$$

   is onto $\mathcal{A}_0$.
4. *If $(\boldsymbol{q},\boldsymbol{\xi})(t)$ is a solution of* (2.19) *in $\mathcal{M}_0$, then $(\boldsymbol{q},\boldsymbol{p})(t) = \boldsymbol{\chi}(\boldsymbol{q},\boldsymbol{\xi})(t)$ is a solution of* (2.4) *with $\lambda(t)$ given by*

$$\lambda(t) = \hat{\lambda}(\boldsymbol{q}(t),\boldsymbol{\xi}(t)) - \frac{d}{dt}\mu(\boldsymbol{q}(t),\boldsymbol{\xi}(t)).$$

The above result shows that, for any smooth function $\hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi})$, the system (2.19) is an ambient-space formulation for (2.4). The physical solution set $\mathcal{M}_0$ for this formulation is an integral level set and hence is neutrally stable; however, its stability can be controlled by introducing a stability parameter α as was done in the last section and an analog of Proposition 2.4 can be obtained.

The function $\hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi})$ can be used to control some aspects of the “non-physical” dynamics in (2.19). For example, the dynamics of the velocity-like variable $m^{-1}\boldsymbol{\xi}$ can be controlled in the direction of $Dg(\boldsymbol{q})$, a direction normal to the physical configuration manifold. In particular, the choice

$$\begin{aligned} \hat{\lambda}(\boldsymbol{q},\boldsymbol{\xi}) = [Dg(\boldsymbol{q})\cdot m^{-1}Dg(\boldsymbol{q})]^{-1}\Big\{ &- Dg(\boldsymbol{q})\cdot m^{-1}DV(\boldsymbol{q}) \\ &+ m^{-1}[\boldsymbol{\xi} - \mu(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q})]\cdot D^2g(\boldsymbol{q})m^{-1}[\boldsymbol{\xi} + \mu(\boldsymbol{q},\boldsymbol{\xi})Dg(\boldsymbol{q})]\Big\} \end{aligned}$$

gives rise to the extra integral $f(\boldsymbol{q},\boldsymbol{\xi}) = Dg(\boldsymbol{q})\cdot m^{-1}\boldsymbol{\xi}$. In infinite dimensions the analog of (2.18) is a boundary value problem, and an extra multiplier field such as λ will provide some flexibility in the consideration of boundary conditions for μ.

3. Infinite-Dimensional Systems

In this section we extend the finite-dimensional results of Section 2 to an infinite-dimensional example. We first establish some notation and then introduce our main model problem: a displacement-traction initial-boundary value problem for a linearized incompressible elastic body in $\mathbb{R}^n$. For more details on the physical and mathematical structure of the field equations of elasticity presented in this section see, for example, [43, 21].

3.1. Preliminaries

Let $C^m(\Omega, F)$ denote the set of all m-times continuously differentiable F-valued functions on an open set Ω in $\mathbb{R}^n$. For $w \in C^m(\Omega, F)$ we define

$$||w||_m = \Big(\sum_{|\sigma| \leqq m} \int_\Omega ||D^\sigma w||^2 \, d\Omega \Big)^{1/2} \quad \text{where} \quad D^\sigma = \frac{\partial^{\sigma_1 + \cdots + \sigma_n}}{\partial X_1^{\sigma_1} \cdots \partial X_n^{\sigma_n}},$$

$\sigma = (\sigma_1, \ldots, \sigma_n)$, $|\sigma| = \sigma_1 + \cdots + \sigma_n$, $(X_1, \ldots, X_n)$ are the coordinates in Ω and $|| \cdot ||$ denotes the norm on F. If we let $\tilde{C}^m(\Omega, F)$ be the subset consisting of those functions w for which $||w||_m < \infty$, then the Hilbert space $H_m(\Omega, F)$ is defined to be the completion of $\tilde{C}^m(\Omega, F)$ in the norm $|| \cdot ||_m$ (see, e.g., [1]). The standard inner-product on $H_m(\Omega, F)$ is defined as

$$\langle v, w \rangle_m = \sum_{|\sigma| \leqq m} \int_\Omega D^\sigma v \cdot D^\sigma w \, d\Omega,$$

where the dot on the right-hand side denotes the standard inner-product on F. For brevity, we shall often omit the subscript m for the case $m = 0$. If we let $\mathring{C}^m(\Omega, F)$ denote the subset consisting of those functions w with compact support, then the completion of $\mathring{C}^m(\Omega, F)$ in the norm $||\cdot||_m$ is a closed subspace of $H_m(\Omega, F)$, which we denote by $\mathring{H}_m(\Omega, F)$. Throughout our developments, Hilbert spaces of the type $H_s(\Omega, F)$ with s non-integer will arise, and these are defined via interpolation, see, e.g., [30]. When there is no danger of confusion, we will sometimes abbreviate $H_m(\Omega, F)$ to H_m.

3.2. Model Problem

In the remainder of our development we will study the linearized equations of motion for an incompressible elastic body. We assume that the body, in its reference or undeformed state, occupies a closed subset $\bar{\Omega}$ of $\mathbb{R}^n$ where Ω is a bounded path-connected open set with piecewise smooth boundary $\partial\Omega$.

We denote the displacement field for the body relative to its reference configuration by $\boldsymbol{u} : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$, and we denote the pressure field by $p : \bar{\Omega} \times [0, T] \to \mathbb{R}$. The pressure field is that part of the total stress field that enforces the linearized incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$, where ∇ denotes the gradient operator relative to the Cartesian coordinates $(X_1, \ldots, X_n)$ in Ω.

The constitutive or active stress field within the body is denoted by $\mathbf{\Sigma} : \bar{\Omega} \times [0, T] \to \mathbb{M}^n$. Here $\mathbb{M}^n$ is the vector space of all real $n \times n$ matrices equipped with the standard (Euclidean) inner product $A : B = A_{ij} B_{ij}$ where summation on repeated indices is implied. In the linearized theory, the active stress $\mathbf{\Sigma}$ is related to the displacement field via a local constitutive relation of the form

$$\mathbf{\Sigma}(\boldsymbol{X}, t) = \mathsf{C}(\boldsymbol{X}, \nabla \boldsymbol{u}(\boldsymbol{X}, t)), \tag{3.1}$$

where $\mathsf{C} : \bar{\Omega} \times \mathbb{M}^n \to \mathbb{M}^n$ is a given field that characterizes the elastic response of the material. For each $\boldsymbol{X} \in \bar{\Omega}$ the mapping $\mathsf{C}(\boldsymbol{X}, \cdot) : \mathbb{M}^n \to \mathbb{M}^n$ is a linear transformation whose kernel contains the set of all skew-symmetric matrices and whose range is contained in the set of all symmetric matrices in $\mathbb{M}^n$. Furthermore, $\mathsf{C}(\boldsymbol{X}, \cdot)$ enjoys the symmetry property

$$A : \mathsf{C}(\boldsymbol{X}, B) = B : \mathsf{C}(\boldsymbol{X}, A) \qquad \forall\, A, B \in \mathbb{M}^n. \tag{3.2}$$

3.2.1. Classic Formulation. Let Γ_σ, Γ_s and Γ_u be disjoint, relatively open subsets of $\partial\Omega$ defined such that $\partial\Omega = \overline{\Gamma_\sigma \cup \Gamma_s \cup \Gamma_u}$. The initial and boundary value problem that we will study is the following:
Find $\boldsymbol{u} : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$ and $p : \bar{\Omega} \times [0, T] \to \mathbb{R}$ such that

$$\begin{aligned}
\rho \ddot{\boldsymbol{u}} &= \nabla \cdot \mathbf{\Sigma} - \nabla p + \boldsymbol{b} && \text{in } \Omega \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega \times [0, T], \\
\mathbf{\Sigma} \boldsymbol{N} - p \boldsymbol{N} &= \mathbf{0} && \text{in } \Gamma_\sigma \times [0, T], \\
\mathbf{\Xi} \mathbf{\Sigma} \boldsymbol{N} &= \mathbf{0} && \text{in } \Gamma_s \times [0, T], \\
\boldsymbol{u} \cdot \boldsymbol{N} &= 0 && \text{in } \Gamma_s \times [0, T], \\
\boldsymbol{u} &= \mathbf{0} && \text{in } \Gamma_u \times [0, T], \\
\boldsymbol{u}(\cdot, 0) &= \hat{\boldsymbol{u}}_0 && \text{in } \bar{\Omega}, \\
\dot{\boldsymbol{u}}(\cdot, 0) &= \hat{\boldsymbol{v}}_0 && \text{in } \bar{\Omega},
\end{aligned} \tag{3.3}$$

where superposed dots denote differentiation with respect to time, $\nabla \cdot \mathbf{\Sigma} : \Omega \times [0, T] \to \mathbb{R}^n$ is defined in components by the relation

$$[\nabla \cdot \mathbf{\Sigma}(\boldsymbol{X}, t)]_i = \partial_j \Sigma_{ij}(X, t), \qquad (i = 1, \ldots, n), \tag{3.4}$$

and ∂_j denotes the partial derivative with respect to the coordinate X_j.

In the above system $\boldsymbol{b}$ is a prescribed body force density per unit reference volume, ρ is the mass density of the body in its reference configuration, $\boldsymbol{N}$ is the unit outward normal field on $\partial\Omega$, $\mathbf{\Xi}$ is a tangential projection field defined at each point of $\partial\Omega$ by the expression $\mathbf{\Xi} = \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}$, $\hat{\boldsymbol{u}}_0$ is a prescribed displacement field, and $\hat{\boldsymbol{v}}_0$ is a prescribed material velocity field.

The partial differential equation $(3.3)_1$ expresses the local balance of linear momentum (whereas the local balance of angular momentum is implied by the symmetry of the total stress field $\mathbf{\Sigma} - p\boldsymbol{I}$), $(3.3)_2$ is the local incompressibility condition, $(3.3)_3$ is a pure traction boundary condition, $(3.3)_{4,5}$ are sliding-type boundary conditions, $(3.3)_6$ is a pure displacement boundary condition, and $(3.3)_{7,8}$ are initial conditions.

Remarks 3.1. 1. There is no real loss of generality in assuming homogeneous boundary data. If the boundary conditions are not homogeneous, then in order to proceed we assume that there exists some function $\tilde{\boldsymbol{u}}$ which satisfies all the boundary conditions. Given $\tilde{\boldsymbol{u}}$, we then solve a homogeneous problem for $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ with $\boldsymbol{b}$ appropriately redefined.

2. The governing equations in $(3.3)_{1,2}$ for $\boldsymbol{u}$ and p may be interpreted as a system of differential-algebraic equations in a function space since no time derivatives of the field p appear; that is, the time evolution of the pressure field is not given explicitly.

3.2.2. Lagrangian Structure. The system in (3.3) has an underlying Lagrangian structure that we will exploit throughout our developments. To bring this structure into evidence, we first introduce the ambient Hilbert space

$$H_1^e = \{\boldsymbol{u} \in H_1 \mid \boldsymbol{u} \cdot \boldsymbol{N} = 0 \quad \text{on} \quad \Gamma_s, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_u\}, \tag{3.5}$$

and a Lagrangian functional $L : H_1^e \times H_0 \to \mathbb{R}$ for the unconstrained elastic body, namely

$$L(\boldsymbol{u}, \dot{\boldsymbol{u}}) = \int_\Omega \tfrac{1}{2}\rho|\dot{\boldsymbol{u}}|^2 - W(\cdot, \nabla\boldsymbol{u})\, d\Omega. \tag{3.6}$$

Here $W : \bar{\Omega} \times \mathbb{M}^n \to \mathbb{R}$ is an energy density function of the form

$$W(\boldsymbol{X}, A) = \tfrac{1}{2} A : \mathsf{C}(\boldsymbol{X}, A) \tag{3.7}$$

where C is the elasticity field of (3.1). The notation $W(\cdot, \nabla\boldsymbol{u})$ is used to denote the function on $\bar{\Omega}$ defined by $\boldsymbol{X} \mapsto W(\boldsymbol{X}, \nabla\boldsymbol{u}(\boldsymbol{X}))$.

We next consider Hamilton's principle for the Lagrangian (3.6) and enforce the pointwise constraint $\nabla \cdot \boldsymbol{u} = 0$ with a multiplier field $p \in H_0$. The associated Euler-Lagrange equations then take the form

$$\begin{aligned} &\frac{d}{dt} D_2\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) \cdot \boldsymbol{\eta} = D_1\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) \cdot \boldsymbol{\eta} && \forall\, \boldsymbol{\eta} \in H_1^e,\ \forall\, t \in (0, T], \\ &D_3\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) \cdot \phi = 0 && \forall\, \phi \in H_0,\ \forall\, t \in [0, T], \end{aligned} \tag{3.8}$$

where $\mathcal{L} : H_1^e \times H_0 \times H_0 \to \mathbb{R}$ is the augmented Lagrangian functional

$$\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) = L(\boldsymbol{u}, \dot{\boldsymbol{u}}) + \int_\Omega p\ \nabla \cdot \boldsymbol{u}\, d\Omega. \tag{3.9}$$

Here $D_1\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) \cdot \boldsymbol{\eta}$ denotes the (partial) directional derivative defined by

$$D_1\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, p) \cdot \boldsymbol{\eta} = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \mathcal{L}(\boldsymbol{u} + \alpha\boldsymbol{\eta}, \dot{\boldsymbol{u}}, p), \tag{3.10}$$

with similar expressions for the other derivatives.

Integrating by parts in (3.8) and introducing the momentum variable $\boldsymbol{\pi} = \rho\dot{\boldsymbol{u}}$ leads to the pointwise equations

$$\begin{aligned}
\dot{\boldsymbol{u}} &= \rho^{-1}\boldsymbol{\pi} && \text{in } \Omega \times (0, T],\\
\dot{\boldsymbol{\pi}} &= \nabla \cdot \boldsymbol{\Sigma} - \nabla p + \boldsymbol{b} && \text{in } \Omega \times (0, T],\\
0 &= \nabla \cdot \boldsymbol{u} && \text{in } \Omega \times [0, T],\\
\boldsymbol{\Sigma N} - p\boldsymbol{N} &= \boldsymbol{0} && \text{in } \Gamma_\sigma \times [0, T],\\
\boldsymbol{\Xi}\boldsymbol{\Sigma N} &= \boldsymbol{0} && \text{in } \Gamma_s \times [0, T],
\end{aligned} \tag{3.11}$$

where the displacement boundary conditions on $\boldsymbol{u}$ are contained in the definition of H_1^e. We interpret (3.11) as a set of evolution equations for $(\boldsymbol{u}, \boldsymbol{\pi}, p)$ in the space $H_1^e \times H_0 \times H_1$, and we typically seek solutions with the following differentiability

$$\begin{aligned}
&\boldsymbol{u} \in \cap_{k=0}^{1} C^{1-k}([0, T], H_{k+1}),\\
&\boldsymbol{\pi} \in \cap_{k=0}^{1} C^{1-k}([0, T], H_k),\\
&p \in C([0, T], H_1).
\end{aligned} \tag{3.12}$$

Notice that solutions of (3.11), if they exist, are restricted to lie in the set

$$\begin{aligned}
\mathcal{A}_0 = \{(\boldsymbol{u}, \boldsymbol{\pi}) \in H_1^e \times H_0 \mid \nabla \cdot \boldsymbol{u} = 0, \quad & \nabla \cdot [\rho^{-1}\boldsymbol{\pi}] = 0\\
\text{and} \quad & \rho^{-1}\boldsymbol{\pi} \cdot \boldsymbol{N}|_{\Gamma_u \cup \Gamma_s} = 0\}.
\end{aligned} \tag{3.13}$$

While an arbitrary element $\rho^{-1}\boldsymbol{\pi} \in H_0$ does not generally have a well defined trace on $\partial\Omega$, we recall that the condition $\nabla \cdot [\rho^{-1}\boldsymbol{\pi}] = 0$ implies $\rho^{-1}\boldsymbol{\pi} \cdot \boldsymbol{N} \in H_{-1/2}(\partial\Omega, \mathbb{R}) = [H_{1/2}(\partial\Omega, \mathbb{R})]'$, thus $\mathcal{A}_0$ is well defined.

3.3. Ambient-Space Formulation: Definition

For a given Hilbert space $\mathcal{H}$, and a given set of boundary conditions, consider a system of evolution equations in the space $H_1^e \times \mathcal{H}$ of the form

$$\begin{aligned}
\dot{\boldsymbol{u}} &= \rho^{-1}\boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi}),\\
\dot{\boldsymbol{\xi}} &= \boldsymbol{w}(\boldsymbol{u}, \boldsymbol{\xi}),
\end{aligned} \tag{3.14}$$

where $\boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi})$ and $\boldsymbol{w}(\boldsymbol{u}, \boldsymbol{\xi})$ are given vector fields that may depend non-locally on $\boldsymbol{u}$ and $\boldsymbol{\xi}$. In exact parallel with the finite-dimensional case, we say that (3.14) is an ambient-space formulation for the constrained system (3.11) if the following conditions hold:

1. The system possesses an invariant set $\mathcal{M}_0 \subset H_1^e \times \mathcal{H}$.
2. The map $\boldsymbol{\chi} : \mathcal{M}_0 \to H_1^e \times H_0$ defined by

$$\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi}) = (\boldsymbol{u}, \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi}))$$

   is onto the set $\mathcal{A}_0$ defined in (3.13).

3. If $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ is a solution in $\mathcal{M}_0$, then $(\boldsymbol{u}, \boldsymbol{\pi})(t) = \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi})(t)$ is a solution of (3.11) for some $p(t)$.

The invariant set $\mathcal{M}_0$ will be called the physical solution set of the ambient-space formulation. Just as in the finite-dimensional case, ambient-space formulations can be constructed for which $\mathcal{M}_0$ is unstable, a level set for a first integral, or exponentially attractive in an appropriate sense.

3.4. Ambient-Space Formulations: Examples

In this section we construct three ambient-space formulations for the constrained system (3.11). We do not pause to justify well-posedness of the formulations, although some remarks along these lines are made throughout. Questions regarding well-posedness are addressed in Section 4 within the context of a specific initial-boundary value problem.

3.4.1. Pressure-Based Formulation. Our first ambient-space formulation follows directly from (3.11) by eliminating the multiplier p. To begin, we formally differentiate the constraint equation $(3.11)_3$ twice with respect to time and substitute from $(3.11)_{1,2}$ to obtain

$$\nabla \cdot [\rho^{-1} \nabla p] = \nabla \cdot [\rho^{-1}(\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b})] \quad \text{in } \Omega. \tag{3.15}$$

We next supplement this equation with two boundary conditions. From $(3.11)_4$ we have

$$p = \boldsymbol{N} \cdot \boldsymbol{\Sigma N} \quad \text{on } \Gamma_\sigma, \tag{3.16}$$

and from $(3.11)_{1,2}$ and the fact that $\boldsymbol{u} \in H_1^e$ we arrive at a condition on $\Gamma_u \cup \Gamma_s$; namely,

$$\nabla p \cdot \boldsymbol{N} = \boldsymbol{N} \cdot [\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b}] \quad \text{on } \Gamma_u \cup \Gamma_s. \tag{3.17}$$

Given the above equations for the pressure field we next consider the following problem: Find $\boldsymbol{u} : [0, T] \to H_1^e$ and $\boldsymbol{\xi} : [0, T] \to H_0$ such that

$$\boxed{\begin{aligned}
&\dot{\boldsymbol{u}} = \rho^{-1}\boldsymbol{\xi} && \text{in } \Omega \times (0, T],\\
&\dot{\boldsymbol{\xi}} = \nabla \cdot \boldsymbol{\Sigma} - \nabla \hat{p}(\boldsymbol{u}) + \boldsymbol{b} && \text{in } \Omega \times (0, T],\\
&\boldsymbol{\Sigma N} - \hat{p}(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{0} && \text{in } \Gamma_\sigma \times [0, T],\\
&\boldsymbol{\Xi \Sigma N} = \boldsymbol{0} && \text{in } \Gamma_s \times [0, T],\\
&\text{where } \hat{p} = \hat{p}(\boldsymbol{u}) \text{ is determined by} &&\\
&\nabla \cdot [\rho^{-1}\nabla \hat{p}] = \nabla \cdot [\rho^{-1}(\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b})] && \text{in } \Omega,\\
&\hat{p} = \boldsymbol{N} \cdot \boldsymbol{\Sigma N} && \text{in } \Gamma_\sigma,\\
&\nabla \hat{p} \cdot \boldsymbol{N} = \boldsymbol{N} \cdot [\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b}] && \text{in } \Gamma_u \cup \Gamma_s.
\end{aligned}} \tag{3.18}$$

The relation between (3.18) and (3.11) is summarized in the following proposition.

Proposition 3.1. *Let* $g : H_1^e \to H_0(\Omega, \mathbb{R})$ *be defined by* $g(\boldsymbol{u}) = \nabla \cdot \boldsymbol{u}$, *let* $f : H_1^e \times H_0 \to H_{-1}(\Omega, \mathbb{R})$ *be defined by* $f(\boldsymbol{u}, \boldsymbol{\xi}) = \nabla \cdot [\rho^{-1}\boldsymbol{\xi}]$, *and for any* $c \in g(H_1^e)$ *let*

$$\mathcal{M}_c = \{(\boldsymbol{u}, \boldsymbol{\xi}) \in H_1^e \times H_0 \mid g(\boldsymbol{u}) = c, \quad f(\boldsymbol{u}, \boldsymbol{\xi}) = 0 \quad \text{and} \quad \rho^{-1}\boldsymbol{\xi} \cdot \boldsymbol{N}|_{\Gamma_u \cup \Gamma_s} = 0\}.$$

Then (3.18) *has the following properties:*

1. *The function* $f(\boldsymbol{u}, \boldsymbol{\xi})$ *is an integral for* (3.18).
2. *The set* $\mathcal{M}_c$ *is an invariant set for* (3.18).
3. *The map* $\boldsymbol{\chi} : \mathcal{M}_0 \to H_1^e \times H_0$ *defined by*

$$\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi}) = (\boldsymbol{u}, \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi})) \quad \text{where} \quad \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi}) = \boldsymbol{\xi}$$

   is onto $\mathcal{A}_0$.
4. *If* $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution of* (3.18) *in* $\mathcal{M}_0$, *then* $(\boldsymbol{u}, \boldsymbol{\pi})(t) = \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution of* (3.11) *with* $p(t)$ *given by*

$$p(t) = \hat{p}(\boldsymbol{u}(t)).$$

The above results show that (3.18) is an ambient-space formulation for (3.11) and follow by direct verification. For example, to see that $f(\boldsymbol{u}, \boldsymbol{\xi})$ is an integral, let $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ be any solution of (3.18) in $H_1^e \times H_0$. Then, using $(3.18)_2$ and the defining equations for $\hat{p}(\boldsymbol{u})$, we have

$$\frac{d}{dt} f(\boldsymbol{u}, \boldsymbol{\xi})(t) = 0,$$

which establishes the first result. The invariance of $\mathcal{M}_c$ follows from the fact that

$$\frac{d}{dt} g(\boldsymbol{u})(t) = f(\boldsymbol{u}, \boldsymbol{\xi})(t), \tag{3.19}$$

and the surjectivity of $\boldsymbol{\chi}$ follows from the fact that $\mathcal{M}_0 = \mathcal{A}_0$ and $\boldsymbol{\chi}$ is the identity on $\mathcal{M}_0$.

Remarks 3.2. 1. The traction boundary condition $(3.18)_3$ can be replaced by its tangential projection $\boldsymbol{\Xi}\boldsymbol{\Sigma}\boldsymbol{N} = \boldsymbol{0}$ in Γ_σ. The reason for this is that the normal component of $(3.18)_3$ appears in $(3.18)_6$.

2. A system analogous to (3.18) is studied in Ebin & Simanca [15] for the free-boundary problem $\Gamma_\sigma = \partial\Omega$, and in the special case of constant density, homogeneous isotropic elastic material law and zero body force. They claim to show that the initial value problem for a system of the form (3.18) is well posed on the physical solution set, that is, when restricted to the invariant set $\mathcal{M}_0 \subset H_1^e \times H_0$. Their study does not address the problem of well-posedness in the ambient space $H_1^e \times H_0$.

The next result, which follows from (3.19) and the fact that $f(\boldsymbol{u}, \boldsymbol{\xi})$ is an integral, shows that the physical solution set $\mathcal{M}_0$ of (3.18) is unstable in an appropriate sense.

Proposition 3.2. *Let $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ be a solution of* (3.18) *with corresponding initial data $(\boldsymbol{u}_0, \boldsymbol{\xi}_0)$ in a neighborhood of $\mathcal{M}_0$. If the initial data satisfy*

$$g(\boldsymbol{u}_0) = a \quad \textit{and} \quad f(\boldsymbol{u}_0, \boldsymbol{\xi}_0) = b$$

for some functions $a, b : \Omega \to \mathbb{R}$, then the solution $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ has the property that

$$g(\boldsymbol{u}(t)) = a + bt \quad \textit{and} \quad f(\boldsymbol{u}(t), \boldsymbol{\xi}(t)) = b.$$

3.4.2. Striction-Based Formulation. As in the finite-dimensional case, our second ambient-space formulation for (3.11) follows from a generalization of the impetus-striction method developed in [31, 11]. First, rather than consider the given constraint $\nabla \cdot \boldsymbol{u} = 0$, we consider the related constraint

$$\nabla \cdot \dot{\boldsymbol{u}} + \alpha \nabla \cdot \boldsymbol{u} = 0, \tag{3.20}$$

where $\alpha \in \mathbb{R}$ is a parameter.

Introducing a multiplier $\mu \in H_0$ for the constraint (3.20) we consider an augmented Lagrangian functional $\mathcal{L} : H_1^e \times H_1^e \times H_0 \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \mu) = \int_\Omega \tfrac{1}{2}\rho|\dot{\boldsymbol{u}}|^2 - W(\cdot, \nabla \boldsymbol{u}) + \boldsymbol{b} \cdot \boldsymbol{u} - \mu[\nabla \cdot \dot{\boldsymbol{u}} + \alpha \nabla \cdot \boldsymbol{u}] \; d\Omega, \tag{3.21}$$

and following [31, 11] we refer to the multiplier μ as the striction field. Substituting the above Lagrangian into the variational principle of Hamilton leads to the Euler-Lagrange equations

$$\begin{aligned} &\frac{d}{dt} D_2\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \mu) \cdot \boldsymbol{\eta} = D_1\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \mu) \cdot \boldsymbol{\eta} && \forall \boldsymbol{\eta} \in H_1^e, \ \forall t \in (0, T], \\ &D_3\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \mu) \cdot \phi = 0 && \forall \phi \in H_0, \ \forall t \in [0, T]. \end{aligned} \tag{3.22}$$

Integrating by parts in (3.22) and introducing the variable $\boldsymbol{\zeta}$ defined by

$$\boldsymbol{\zeta} = \rho \, \dot{\boldsymbol{u}} + \nabla \mu \tag{3.23}$$

leads to the equations

$$\begin{aligned} \dot{\boldsymbol{u}} &= \rho^{-1}[\boldsymbol{\zeta} - \nabla \mu] && \text{in } \Omega \times (0, T], \\ \dot{\boldsymbol{\zeta}} &= \nabla \cdot \boldsymbol{\Sigma} + \alpha \nabla \mu + \boldsymbol{b} && \text{in } \Omega \times (0, T], \\ 0 &= \nabla \cdot [\rho^{-1}(\boldsymbol{\zeta} - \nabla \mu)] + \alpha \nabla \cdot \boldsymbol{u} && \text{in } \Omega \times [0, T], \\ \boldsymbol{\Sigma N} - (\dot{\mu} - \alpha\mu)\boldsymbol{N} &= \boldsymbol{0} && \text{in } \Gamma_\sigma \times [0, T], \\ \boldsymbol{\Xi \Sigma N} &= \boldsymbol{0} && \text{in } \Gamma_s \times [0, T]. \end{aligned} \tag{3.24}$$

To develop an ambient-space formulation we next solve for the striction μ as a function of the state variables $\boldsymbol{u}$ and $\boldsymbol{\zeta}$. Using $(3.24)_3$ we get the equation

$$\nabla \cdot [\rho^{-1} \nabla \mu] = \nabla \cdot [\rho^{-1} \boldsymbol{\zeta} + \alpha \boldsymbol{u}] \quad \text{in } \Omega, \tag{3.25}$$

to which we must append appropriate boundary conditions in order to uniquely determine μ. From $(3.24)_1$ and the fact that $\boldsymbol{u} \in H_1^e$ we arrive at a Neumann-type condition on $\Gamma_u \cup \Gamma_s$; namely,

$$\nabla \mu \cdot \boldsymbol{N} = \boldsymbol{\zeta} \cdot \boldsymbol{N} \quad \text{on } \Gamma_u \cup \Gamma_s. \tag{3.26}$$

For the portion Γ_σ we note that a Dirichlet-type boundary condition is provided by the evolution equation

$$\dot{\mu} = \boldsymbol{N} \cdot \boldsymbol{\Sigma} \boldsymbol{N} + \alpha \mu \quad \text{on } \Gamma_\sigma \tag{3.27}$$

which is derived from $(3.24)_4$. Introducing a boundary density ν via the equation

$$\nu = \mu \qquad \text{on } \Gamma_\sigma, \tag{3.28}$$

we consider the following problem: Find $\boldsymbol{u} : [0, T] \to H_1^e$ and $\boldsymbol{\xi} = (\boldsymbol{\zeta}, \nu) : [0, T] \to H_0 \times H_{1/2}(\Gamma_\sigma, \mathbb{R})$ such that

$$\boxed{\begin{array}{ll}
\dot{\boldsymbol{u}} = \rho^{-1}[\boldsymbol{\zeta} - \nabla \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)] & \text{in } \Omega \times (0, T], \\
\dot{\boldsymbol{\zeta}} = \nabla \cdot \boldsymbol{\Sigma} + \alpha \nabla \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) + \boldsymbol{b} & \text{in } \Omega \times (0, T], \\
\dot{\nu} = \boldsymbol{N} \cdot \boldsymbol{\Sigma} \boldsymbol{N} + \alpha \nu & \text{in } \Gamma_\sigma \times [0, T], \\
\boldsymbol{\Xi} \boldsymbol{\Sigma} \boldsymbol{N} = \boldsymbol{0} & \text{in } \Gamma_\sigma \times [0, T], \\
\boldsymbol{\Xi} \boldsymbol{\Sigma} \boldsymbol{N} = \boldsymbol{0} & \text{in } \Gamma_s \times [0, T], \\
\text{where } \mu = \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) \text{ is determined by} & \\
\nabla \cdot [\rho^{-1} \nabla \mu] = \nabla \cdot [\rho^{-1} \boldsymbol{\zeta} + \alpha \boldsymbol{u}] & \text{in } \Omega, \\
\mu = \nu & \text{in } \Gamma_\sigma, \\
\nabla \mu \cdot \boldsymbol{N} = \boldsymbol{\zeta} \cdot \boldsymbol{N} & \text{in } \Gamma_u \cup \Gamma_s.
\end{array}} \tag{3.29}$$

The relation between (3.29) and (3.11) is summarized in the following proposition.

Proposition 3.3. *Let $g : H_1^e \to H_0(\Omega, \mathbb{R})$ be as above and for any $c \in g(H_1^e)$ let $\mathcal{M}_c = \{(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) \in H_1^e \times H_0 \times H_{1/2} \mid g(\boldsymbol{u}) = c\}$. Then* (3.29) *has the following properties:*

1. *The function $g(\boldsymbol{u})$ is an integral for* (3.29) *if $\alpha = 0$.*
2. *The set $\mathcal{M}_0$ is an invariant set for* (3.29) *for any $\alpha \in \mathbb{R}$.*
3. *The map $\boldsymbol{\chi} : \mathcal{M}_0 \to H_1^e \times H_0$ defined by*

$$\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) = (\boldsymbol{u}, \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)) \quad \textit{where} \quad \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) = \boldsymbol{\zeta} - \nabla \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)$$

   is onto $\mathcal{A}_0$.
4. *If $(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t)$ is a solution of* (3.29) *in $\mathcal{M}_0$, then $(\boldsymbol{u}, \boldsymbol{\pi}) = \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)$ is a solution of* (3.11) *with $p(t)$ given by*

$$p(t) = \frac{d}{dt} \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t) - \alpha \mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t).$$

To establish the first result let $(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t)$ be an arbitrary solution of (3.29) in $H_1^e \times H_0 \times H_{1/2}$. Then

$$\frac{d}{dt} g(\boldsymbol{u})(t) = \nabla \cdot \dot{\boldsymbol{u}}(t) = -\alpha \nabla \cdot \boldsymbol{u}(t) = -\alpha g(\boldsymbol{u})(t). \tag{3.30}$$

If $\alpha = 0$, we deduce that $g(\boldsymbol{u})$ is constant along arbitrary solutions, and the result follows. Integrating (3.30) yields

$$g(\boldsymbol{u})(t) = e^{-\alpha t} g(\boldsymbol{u})(0), \tag{3.31}$$

and the second result follows upon noting that any solution with initial data in $\mathcal{M}_0$ satisfies $g(\boldsymbol{u})(0) = 0$. To establish the third result, consider an arbitrary element $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\pi}}) \in \mathcal{A}_0$, so $\nabla \cdot \bar{\boldsymbol{u}} = 0$, $\nabla \cdot [\rho^{-1}\bar{\boldsymbol{\pi}}] = 0$ and $\bar{\boldsymbol{\pi}} \cdot \boldsymbol{N}|_{\Gamma_u \cup \Gamma_s} = 0$. Taking $\boldsymbol{u} = \bar{\boldsymbol{u}}$, $\boldsymbol{\zeta} = \bar{\boldsymbol{\pi}}$ and $\nu = 0$ we find $\mu(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) = 0$, and the result follows since $(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) \in \mathcal{M}_0$ and $\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\zeta}, \nu) = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{\pi}})$. The fourth result follows by direct verification.

The above results show that (3.29) is an ambient-space formulation for (3.11). The next proposition summarizes the stability properties of the physical solution set $\mathcal{M}_0$ of (3.29); in particular, the set $\mathcal{M}_0$ is exponentially attractive in an appropriate sense if $\alpha > 0$.

Proposition 3.4. *Let $(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t)$ be a solution of* (3.29) *with corresponding initial data $(\boldsymbol{u}_0, \boldsymbol{\zeta}_0, \nu_0)$ in a neighborhood of $\mathcal{M}_0$. If the data satisfy $g(\boldsymbol{u}_0) = a$ for some function $a : \Omega \to \mathbb{R}$, then the solution $(\boldsymbol{u}, \boldsymbol{\zeta}, \nu)(t)$ has the property that $g(\boldsymbol{u}(t)) = ae^{-\alpha t}$.*

Remark 3.3. For $\alpha = 0$, the formulation in (3.29) possesses a Hamiltonian structure that can be developed by employing a slight generalization of the impetus-striction formalism in [11,31]. As compared to those in [11,31], the above formulation is more complicated in the sense that the impetus variable $\boldsymbol{\xi} = (\boldsymbol{\zeta}, \nu)$ has two components: a bulk component $\boldsymbol{\zeta}$ defined in the interior of the spatial domain and a singular component ν defined on a portion of the boundary. In accordance with the original formalism, the striction μ is defined at any instant in time through a minimization, but here one of the boundary conditions for the striction is governed by an explicit evolution equation defined on the boundary. These complications do not arise in the example of [11] because only one space dimension is considered, and do not arise in the example of [31] because of the specific boundary conditions appropriate for inviscid fluid flow.

3.4.3. Pressure-Striction Formulation. In this section we use a multi-multiplier approach to construct an ambient space formulation without boundary evolution equations. As shown earlier, a formulation based on a single pressure-type multiplier does not involve boundary evolution equations; however, its physical solution set is unstable. By employing different types of multipliers we will be able to avoid evolution equations on the boundary without sacrificing the stability of the physical solution set.

Motivated as in Section 2 within the finite-dimensional case, we introduce a multiplier $\lambda \in H_0$ for the displacement-level constraint $\nabla \cdot \boldsymbol{u} = 0$ and a multiplier

$\mu \in H_0$ for the velocity-level constraint $\nabla \cdot \dot{\boldsymbol{u}} = 0$, and we consider an augmented Lagrangian $\mathcal{L} : H_1^e \times H_1^e \times H_0 \times H_0 \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \lambda, \mu) = \int_\Omega \tfrac{1}{2}\rho|\dot{\boldsymbol{u}}|^2 - W(\cdot, \nabla\boldsymbol{u}) + \boldsymbol{b}\cdot\boldsymbol{u} + \lambda\,\nabla\cdot\boldsymbol{u} - \mu\,\nabla\cdot\dot{\boldsymbol{u}}\;\; d\Omega. \quad (3.32)$$

Notice that the multiplier λ is a pressure-like variable while the multiplier μ is striction-like. Substituting the above Lagrangian into the variational principle of Hamilton leads to the Euler-Lagrange equations

$$\begin{aligned} &\frac{d}{dt}D_2\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \lambda, \mu)\cdot\boldsymbol{\eta} = D_1\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \lambda, \mu)\cdot\boldsymbol{\eta} && \forall\boldsymbol{\eta}\in H_1^e, \forall t\in(0,T],\\ &D_3\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \lambda, \mu)\cdot\phi = 0 && \forall\phi\in H_0, \forall t\in[0,T],\\ &D_4\mathcal{L}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \lambda, \mu)\cdot\phi = 0 && \forall\phi\in H_0,\ \forall t\in[0,T]. \end{aligned} \quad (3.33)$$

Integrating by parts in (3.33) and introducing the variable $\boldsymbol{\xi}$ defined by

$$\boldsymbol{\xi} = \rho\,\dot{\boldsymbol{u}} + \nabla\mu \quad (3.34)$$

leads to the equations

$$\begin{aligned} \dot{\boldsymbol{u}} &= \rho^{-1}[\boldsymbol{\xi} - \nabla\mu] && \text{in } \Omega\times(0,T],\\ \dot{\boldsymbol{\xi}} &= \nabla\cdot\boldsymbol{\Sigma} - \nabla\lambda + \boldsymbol{b} && \text{in } \Omega\times(0,T],\\ 0 &= \nabla\cdot\boldsymbol{u} && \text{in } \Omega\times[0,T],\\ 0 &= \nabla\cdot[\rho^{-1}(\boldsymbol{\xi} - \nabla\mu)] && \text{in } \Omega\times[0,T],\\ \boldsymbol{\Sigma}\boldsymbol{N} - (\lambda + \dot{\mu})\boldsymbol{N} &= \boldsymbol{0} && \text{in } \Gamma_\sigma\times[0,T],\\ \boldsymbol{\Xi}\,\boldsymbol{\Sigma}\boldsymbol{N} &= \boldsymbol{0} && \text{in } \Gamma_s\times[0,T]. \end{aligned} \quad (3.35)$$

To develop an ambient-space formulation, we proceed as in the finite-dimensional case and solve for the striction μ as a function of the state variables $\boldsymbol{u}$ and $\boldsymbol{\xi}$ while leaving λ arbitrary. Using $(3.35)_4$ we get the equation

$$\nabla\cdot[\rho^{-1}\nabla\mu] = \nabla\cdot[\rho^{-1}\boldsymbol{\xi}] \quad \text{in } \Omega, \quad (3.36)$$

to which we must append appropriate boundary conditions in order to uniquely determine μ. From $(3.35)_1$ and the fact that $\boldsymbol{u} \in H_1^e$ we arrive at a Neumann-type condition on $\Gamma_u \cup \Gamma_s$; namely,

$$\nabla\mu\cdot\boldsymbol{N} = \boldsymbol{\xi}\cdot\boldsymbol{N} \quad \text{on } \Gamma_u\cup\Gamma_s. \quad (3.37)$$

For the portion Γ_σ we note that we can avoid the evolution term in $(3.35)_5$ by imposing the condition

$$\mu = 0 \quad \text{on } \Gamma_\sigma. \quad (3.38)$$

This choice for μ implies that the arbitrary multiplier field λ must now satisfy the boundary condition

$$\lambda = \boldsymbol{N}\cdot\boldsymbol{\Sigma}\boldsymbol{N} \quad \text{on } \Gamma_\sigma. \quad (3.39)$$

As we remarked in Section 2, the multiplier λ can be used to control certain aspects of the "non-physical" dynamics in an ambient-space formulation. For example, if λ satisfies the equation

$$\nabla \cdot [\rho^{-1}\nabla\lambda] = \nabla \cdot [\rho^{-1}(\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b})] \quad \text{in } \Omega, \tag{3.40}$$

then the function $f(\boldsymbol{u}, \boldsymbol{\xi}) = \nabla \cdot [\rho^{-1}\boldsymbol{\xi}]$ will be an integral in the resulting formulation. This choice of λ thus controls the divergence of the velocity-like variable $\rho^{-1}\boldsymbol{\xi}$. To uniquely specify λ it remains to specify boundary conditions on the portion $\Gamma_u \cup \Gamma_s$, and to this end we specify natural boundary conditions associated with a weak formulation of (3.40), namely

$$\nabla\lambda \cdot \boldsymbol{N} = (\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b}) \cdot \boldsymbol{N} \quad \text{on } \Gamma_u \cup \Gamma_s. \tag{3.41}$$

Given the above expressions for μ and λ we consider the following problem: Find $\boldsymbol{u} : [0, T] \to H_1^e$ and $\boldsymbol{\xi} : [0, T] \to H_0$ such that

$$\boxed{\begin{array}{ll}
\dot{\boldsymbol{u}} = \rho^{-1}[\boldsymbol{\xi} - \nabla\mu(\boldsymbol{\xi})] & \text{in } \Omega \times (0, T], \\
\dot{\boldsymbol{\xi}} = \nabla \cdot \boldsymbol{\Sigma} - \nabla\lambda(\boldsymbol{u}) + \boldsymbol{b} & \text{in } \Omega \times (0, T], \\[1ex]
\boldsymbol{\Sigma}\boldsymbol{N} - \lambda(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{0} & \text{in } \Gamma_\sigma \times [0, T], \\
\boldsymbol{\Xi}\,\boldsymbol{\Sigma}\boldsymbol{N} = \boldsymbol{0} & \text{in } \Gamma_s \times [0, T], \\[1ex]
\text{where } \mu = \mu(\boldsymbol{\xi}) \text{ and } \lambda = \lambda(\boldsymbol{u}) \text{ are determined by} & \\[1ex]
\nabla \cdot [\rho^{-1}\nabla\mu] = \nabla \cdot [\rho^{-1}\boldsymbol{\xi}] & \text{in } \Omega, \\
\mu = 0 & \text{in } \Gamma_\sigma, \\
\nabla\mu \cdot \boldsymbol{N} = \boldsymbol{\xi} \cdot \boldsymbol{N} & \text{in } \Gamma_u \cup \Gamma_s, \\[1ex]
\nabla \cdot [\rho^{-1}\nabla\lambda] = \nabla \cdot [\rho^{-1}(\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b})] & \text{in } \Omega, \\
\lambda = \boldsymbol{N} \cdot \boldsymbol{\Sigma}\boldsymbol{N} & \text{in } \Gamma_\sigma, \\
\nabla\lambda \cdot \boldsymbol{N} = (\nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{b}) \cdot \boldsymbol{N} & \text{in } \Gamma_u \cup \Gamma_s.
\end{array}} \tag{3.42}$$

The relation between (3.42) and (3.11) is summarized in the following proposition.

Proposition 3.5. *Let $g : H_1^e \to H_0(\Omega, \mathbb{R})$ be as before and for any $c \in g(H_1^e)$ let $\mathcal{M}_c = \{(\boldsymbol{u}, \boldsymbol{\xi}) \in H_1^e \times H_0 \mid g(\boldsymbol{u}) = c\}$. Then* (3.42) *has the following properties:*

1. *The function $g(\boldsymbol{u})$ is an integral for* (3.42)*, thus $\mathcal{M}_0$ is invariant.*
2. *The map $\boldsymbol{\chi} : \mathcal{M}_0 \to H_1^e \times H_0$ defined by*

$$\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi}) = (\boldsymbol{u}, \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi})) \quad \textit{where} \quad \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi}) = \boldsymbol{\xi} - \nabla\mu(\boldsymbol{\xi})$$

is onto $\mathcal{A}_0$.

3. *If* $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution of* (3.42) *in* $\mathcal{M}_0$, *then* $(\boldsymbol{u}, \boldsymbol{\pi})(t) = \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution of* (3.11) *with the pressure* $p(t)$ *given by*

$$p(t) = \lambda(\boldsymbol{u})(t) + \frac{d}{dt}\mu(\boldsymbol{\xi})(t).$$

The above results show that (3.42) is an ambient-space formulation for (3.11) and follow by direct verification. For example, to see that $g(\boldsymbol{u})$ is an integral, let $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ be any solution of (3.42) in $H_1^e \times H_0$. Then, using $(3.42)_1$ and the defining equations for $\mu(\boldsymbol{\xi})$, we have

$$\frac{d}{dt} g(\boldsymbol{u})(t) = 0,$$

which establishes the first result. To establish the surjectivity of $\boldsymbol{\chi}$ consider an arbitrary element $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\pi}}) \in \mathcal{A}_0$, so $\nabla \cdot \bar{\boldsymbol{u}} = 0$, $\nabla \cdot [\rho^{-1}\bar{\boldsymbol{\pi}}] = 0$ and $\bar{\boldsymbol{\pi}} \cdot \boldsymbol{N}|_{\Gamma_u \cup \Gamma_s} = 0$. Taking $\boldsymbol{u} = \bar{\boldsymbol{u}}$ and $\boldsymbol{\xi} = \bar{\boldsymbol{\pi}}$ we find $\mu(\boldsymbol{\xi}) = 0$, and the result follows since $(\boldsymbol{u}, \boldsymbol{\xi}) \in \mathcal{M}_0$ and $\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi}) = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{\pi}})$.

In contrast with the pure pressure formulation, the physical solution set $\mathcal{M}_0$ of (3.42) is neutrally stable since it may be interpreted as the level set of the integral $g(\boldsymbol{u})$. However, the stability properties of $\mathcal{M}_0$ in the present case can be enhanced by appropriately changing the boundary value problem for the striction μ. To this end, consider (3.42) with the following modified boundary value problem for the striction μ

$$\begin{aligned} \nabla \cdot [\rho^{-1}\nabla\mu] &= \nabla \cdot [\rho^{-1}\boldsymbol{\xi}] + \alpha\nabla \cdot \boldsymbol{u} && \text{in } \Omega, \\ \mu &= 0 && \text{in } \Gamma_\sigma, \\ \nabla\mu \cdot \boldsymbol{N} &= \boldsymbol{\xi} \cdot \boldsymbol{N} && \text{in } \Gamma_u \cup \Gamma_s, \end{aligned} \tag{3.43}$$

where $\alpha \in \mathbb{R}$ is a parameter. The relation between (3.42) subject to (3.43) and (3.11) is summarized in the following proposition which may be readily verified.

Proposition 3.6. *Consider* (3.42) *subject to* (3.43) *and let* $g(\boldsymbol{u})$ *and* $\mathcal{M}_c$ *be as in Proposition 3.5. Then* (3.42) *has the following properties:*

1. *The set* $\mathcal{M}_0$ *is invariant for any fixed* $\alpha \in \mathbb{R}$.
2. *The map* $\boldsymbol{\chi} : \mathcal{M}_0 \to H_1^e \times H_0$ *defined by*

$$\boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi}) = (\boldsymbol{u}, \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi})) \quad \textit{where} \quad \boldsymbol{\pi}(\boldsymbol{u}, \boldsymbol{\xi}) = \boldsymbol{\xi} - \nabla\mu(\boldsymbol{\xi})$$

is onto $\mathcal{A}_0$.

3. *If* $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution in* $\mathcal{M}_0$, *then* $(\boldsymbol{u}, \boldsymbol{\pi})(t) = \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{\xi})(t)$ *is a solution of* (3.11) *with* $p(t)$ *given by*

$$p(t) = \lambda(\boldsymbol{u})(t) + \frac{d}{dt}\mu(\boldsymbol{u}, \boldsymbol{\xi})(t).$$

4. *The set* $\mathcal{M}_0$ *is exponentially attractive for* $\alpha > 0$ *in the following sense: any solution of* (3.42) *subject to* (3.43) *has the property*

$$g(\boldsymbol{u})(t) = e^{-\alpha t} g(\boldsymbol{u})(0).$$

4. Existence and Uniqueness Results

In this section we state a well-posedness result for the pressure-striction ambient-space formulation outlined in Section 3.4.3. We consider the inhomogeneous Neumann problem defined by the conditions $\Gamma_u = \emptyset$, $\Gamma_s = \emptyset$ and $\Gamma_\sigma = \Gamma$, and consider a homogeneous, isotropic material.

In proving the well-posedness of our ambient-space formulation we prove existence and uniqueness for an inhomogeneous Neumann problem in linearized incompressible elastodynamics. For compressible elastodynamics we note that much work has been done within both the linear and nonlinear settings. Various results for the linear case are reviewed in [32], and results for the nonlinear case have appeared more recently. For example, the initial value problem posed on all of space is considered in [22], the Dirichlet initial-boundary value problem is considered in [25, 7, 10] and the Neumann initial-boundary value problem in two space dimensions is considered in [38].

For incompressible elastodynamics, the initial value problem for nonlinear materials posed on all of space is treated in [14, 17] and the Dirichlet initial-boundary value problem is treated in [23]. The Neumann initial-boundary value problem for the linearized case is treated in [15] and was subsequently extended to the nonlinear case in [16]. Here we note that the aforementioned results for the incompressible case have been restricted to constrained formulations of the problem. A different approach to the initial value problem posed in all of space was taken in [39], where results for the incompressible case were established by passing to a limit from the compressible case.

Our well-posedness result for linearized incompressible elastodynamics, contained in Theorem 4.1 below, generalizes a result claimed by EBIN & SIMANCA [15] who studied a different formulation of the same physical problem. (The weak formulation given in [15] is not correct, actually. Proposition 3.17 in [15] is false, for example.) In [15] the system of interest is formulated only for solutions that satisfy the configuration constraints. In contrast, the system considered here is formulated in an ambient space, which yields information regarding the stability of the formulation to perturbations that fail to respect the physical constraints.

4.1. The Inhomogeneous Neumann Problem

The problem we will study is that of finding a displacement field $\boldsymbol{u} : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$ and an impetus field $\boldsymbol{\xi} : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$ such that

$$
\boxed{
\begin{array}{ll}
\dot{\boldsymbol{u}} = \rho^{-1}[\boldsymbol{\xi} - \nabla\mu(\boldsymbol{\xi})] & \text{in } \Omega \times (0, T], \\
\dot{\boldsymbol{\xi}} = \nabla \cdot \boldsymbol{\Sigma} - \nabla\lambda(\boldsymbol{u}) + \boldsymbol{b} & \text{in } \Omega \times (0, T], \\
\boldsymbol{\Sigma N} - \lambda(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{h} & \text{in } \Gamma \times [0, T], \\
\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 & \text{in } \bar{\Omega}, \\
\boldsymbol{\xi}(\cdot, 0) = \boldsymbol{\xi}_0 & \text{in } \bar{\Omega}, \\
\nabla \cdot [\rho^{-1}\nabla\mu] = \nabla \cdot [\rho^{-1}\boldsymbol{\xi}] & \text{in } \Omega, \\
\qquad \mu = 0 & \text{on } \Gamma, \\
\nabla \cdot [\rho^{-1}\nabla\lambda] = \nabla \cdot [\rho^{-1}\nabla \cdot \boldsymbol{\Sigma} + \rho^{-1}\boldsymbol{b}] & \text{in } \Omega, \\
\qquad \lambda = \boldsymbol{N} \cdot \boldsymbol{\Sigma N} - \boldsymbol{N} \cdot \boldsymbol{h} & \text{on } \Gamma.
\end{array}
}
\tag{4.1}
$$

For the case of a homogeneous, isotropic material we suppose that the mass density field $\rho > 0$ is constant and that the constitutive stress field $\boldsymbol{\Sigma}$ is of the form

$$
\boldsymbol{\Sigma} = \mathsf{C}(\nabla \boldsymbol{u}) = 2\vartheta \, \mathrm{sym}[\nabla \boldsymbol{u}] \tag{4.2}
$$

where $\vartheta > 0$ is the constant shear modulus and sym$[\cdot]$ is the symmetric projection on $\mathbb{M}^n$.

Remark 4.1. The traction boundary condition $(4.1)_3$ can be replaced by its tangential projection $\boldsymbol{\Xi}\,\boldsymbol{\Sigma N} = \boldsymbol{\Xi}\boldsymbol{h}$ in Γ. The reason for this is that the normal component of $(4.1)_3$ appears in $(4.1)_9$.

4.2. Abstract Formulation and Main Result

In this subsection we set up a Hilbert space formulation of the system in (4.1) and state the well-posedness result. To begin, we introduce the Hilbert spaces

$$
\begin{aligned}
\mathcal{V} &= H_1(\Omega, \mathbb{R}^n), \\
\mathcal{W} &= H_0(\Omega, \mathbb{R}^n),
\end{aligned}
$$

equipped with the inner-products

$$
\begin{aligned}
\langle \cdot, \cdot \rangle_{\mathcal{V}} &= \alpha \langle \cdot, \rho\, \cdot \rangle_0 + \langle \nabla(\cdot), \mathsf{C}\nabla(\cdot) \rangle_0, \\
\langle \cdot, \cdot \rangle_{\mathcal{W}} &= \left\langle \cdot, \rho^{-1} \cdot \right\rangle_0,
\end{aligned}
$$

where $\alpha \geqq 0$ is a suitable constant. For $\vartheta > 0$ the elasticity field C satisfies a strong ellipticity condition and thus there is an α for which the inner-product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ is equivalent to $\langle \cdot, \cdot \rangle_1$ (see, e.g., [32,18]). Recall that equivalence of the inner products implies there is a constant $d > 0$ such that

$$
\frac{1}{d^2} ||\boldsymbol{v}||_1^2 \leqq \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{V}} := ||\boldsymbol{v}||_{\mathcal{V}}^2 \leqq d^2 ||\boldsymbol{v}||_1^2 \quad \forall\, \boldsymbol{v} \in \mathcal{V}.
$$

Similarly, the inner-product $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ is equivalent to $\langle\cdot,\cdot\rangle_0$.

To develop an abstract formulation of (4.1) we introduce the following operators:

$$\begin{aligned}
&\boldsymbol{B} : D(\boldsymbol{B}) \subset \mathcal{V} \to \mathcal{W},\\
&D(\boldsymbol{B}) := \{\boldsymbol{u} \in \mathcal{V} \mid \boldsymbol{u} \in H_2(\Omega, \mathbb{R}^n),\ \ \boldsymbol{\Xi}\boldsymbol{\Sigma}(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{\Xi}\boldsymbol{h} \text{ on } \Gamma\},\\
&\boldsymbol{B}(\boldsymbol{u}) := \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{u}) - \nabla\lambda(\boldsymbol{u}),\\[1ex]
&\boldsymbol{C} : D(\boldsymbol{C}) \subset \mathcal{W} \to \mathcal{V},\\
&D(\boldsymbol{C}) := \{\boldsymbol{\xi} \in \mathcal{W} \mid \boldsymbol{\xi} \in H_1(\Omega, \mathbb{R}^n)\},\\
&\boldsymbol{C}(\boldsymbol{\xi}) := \rho^{-1}[\boldsymbol{\xi} - \nabla\mu(\boldsymbol{\xi})],\\[1ex]
&\boldsymbol{A} : D(\boldsymbol{A}) \subset \mathcal{V}\times\mathcal{W} \to \mathcal{V}\times\mathcal{W},\\
&D(\boldsymbol{A}) := D(\boldsymbol{B}) \times D(\boldsymbol{C}),\\
&\boldsymbol{A}(\boldsymbol{u},\boldsymbol{\xi}) := (\boldsymbol{C}(\boldsymbol{\xi}), \boldsymbol{B}(\boldsymbol{u})).
\end{aligned} \tag{4.3}$$

Remarks 4.2. 1. For the operator $\boldsymbol{B}$ to be well defined it is required that $\lambda(\boldsymbol{u})$ be in $H_1(\Omega,\mathbb{R})$ given that $\boldsymbol{u}$ is in $H_2(\Omega,\mathbb{R}^n)$. To see that this is indeed the case, assume $\boldsymbol{b}$ is in $H_0(\Omega,\mathbb{R}^n)$ and that $\boldsymbol{h}$ is in $H_{\frac{1}{2}}(\Gamma,\mathbb{R}^n)$. Then the equation

$$\left\langle\nabla\bar{\lambda}, \rho^{-1}\nabla\phi\right\rangle_0 = \left\langle\nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{u}) + \boldsymbol{b}, \rho^{-1}\nabla\phi\right\rangle_0 \quad \forall\phi\in\mathring{H}_1(\Omega,\mathbb{R}) \tag{4.4}$$

has a unique solution $\bar{\lambda}(\boldsymbol{u}) \in \mathring{H}_1(\Omega,\mathbb{R})$, and the equations

$$\begin{aligned}
\nabla\cdot[\rho^{-1}\nabla\tilde{\lambda}] &= 0 && \text{in } \Omega,\\
\tilde{\lambda} &= \boldsymbol{N}\cdot\boldsymbol{\Sigma}(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{N}\cdot\boldsymbol{h} && \text{on } \Gamma
\end{aligned} \tag{4.5}$$

have a unique solution $\tilde{\lambda}(\boldsymbol{u}) \in H_1(\Omega,\mathbb{R})$ (see, e.g., [34, 30]). Setting $\lambda = \bar{\lambda} + \tilde{\lambda}$ we see that λ satisfies $(4.1)_{8,9}$ and $\lambda \in H_1(\Omega,\mathbb{R})$.

2. For any $\phi \in \mathring{H}_1(\Omega,\mathbb{R})$ we have

$$\left\langle\boldsymbol{B}(\boldsymbol{u}) + \boldsymbol{b}, \rho^{-1}\nabla\phi\right\rangle_0 = 0. \tag{4.6}$$

This follows from (4.4), (4.5) and the decomposition $\lambda = \bar{\lambda} + \tilde{\lambda}$.

3. For the operator $\boldsymbol{C}$ to be well defined it is required that $\mu(\boldsymbol{\xi})$ be in $H_2(\Omega,\mathbb{R})$ given that $\boldsymbol{\xi}$ is in $H_1(\Omega,\mathbb{R}^n)$. This follows from $(4.1)_{6,7}$ and the standard theory for such equations (see, e.g., [34, 30]).

4. Note that $\nabla\cdot\boldsymbol{C}(\boldsymbol{\xi}) = 0$ for any $\boldsymbol{\xi} \in D(\boldsymbol{C})$.

An abstract formulation of (4.1) can be stated as follows:

Given $\boldsymbol{b} \in H_0(\Omega, \mathbb{R}^n)$ and $\boldsymbol{h} \in H_{\frac{1}{2}}(\Gamma, \mathbb{R}^n)$, and initial data $(\boldsymbol{u}_0, \boldsymbol{\xi}_0) \in D(\boldsymbol{A})$, find a curve $(\boldsymbol{u}, \boldsymbol{\xi}) : [0, T] \to D(\boldsymbol{A}) \subset \mathcal{V} \times \mathcal{W}$ such that $(\boldsymbol{u}, \boldsymbol{\xi})(0) = (\boldsymbol{u}_0, \boldsymbol{\xi}_0)$ and

$$\begin{Bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\xi}} \end{Bmatrix} = \boldsymbol{A} \begin{Bmatrix} \boldsymbol{u} \\ \boldsymbol{\xi} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{Bmatrix} \qquad \forall t \in (0, T]. \tag{4.7}$$

Our main result on the existence and uniqueness of solutions to (4.7) is contained in the following statement.

Theorem 4.1. *Assume the domain Ω is open, bounded and of class C^2. Then, given $\boldsymbol{b} \in H_0(\Omega, \mathbb{R}^n)$, $\boldsymbol{h} \in H_{\frac{1}{2}}(\Gamma, \mathbb{R}^n)$ and initial data $(\boldsymbol{u}_0, \boldsymbol{\xi}_0) \in D(\boldsymbol{A})$, there is a unique continuously differentiable curve $(\boldsymbol{u}, \boldsymbol{\xi}) : [0, T] \to D(\boldsymbol{A}) \subset \mathcal{V} \times \mathcal{W}$ satisfying* (4.7) *and* $(\boldsymbol{u}, \boldsymbol{\xi})(0) = (\boldsymbol{u}_0, \boldsymbol{\xi}_0)$.

5. Proof of Well-Posedness

In this section we prove Theorem 4.1. We first introduce an auxiliary problem and show that, for appropriate inhomogeneous data, solutions of this problem satisfy (4.7). We then use the theory of semigroups to establish existence and uniqueness of solutions to the auxiliary problem with homogeneous data. This result is then extended to inhomogeneous data by use of a variation of constants formula, and the result for (4.7) will follow.

5.1. An Auxiliary Problem

In this subsection we introduce an auxiliary problem that will prove useful in the analysis of (4.7). To begin, we define closed subspaces $\mathcal{V}_{\text{div}}$ and $\mathcal{W}_{\text{div}}$ as

$$\begin{aligned} \mathcal{V}_{\text{div}} &:= \{\boldsymbol{v} \in \mathcal{V} \mid \nabla \cdot \boldsymbol{v} = 0\}, \\ \mathcal{W}_{\text{div}} &:= \{\boldsymbol{w} \in \mathcal{W} \mid \left\langle \boldsymbol{w}, \rho^{-1}\nabla\phi \right\rangle_0 = 0 \quad \forall \phi \in \mathring{H}_1(\Omega, \mathbb{R})\}, \end{aligned} \tag{5.1}$$

and introduce operators $\boldsymbol{B}_{\text{div}}$, $\boldsymbol{C}_{\text{div}}$ and $\boldsymbol{A}_{\text{div}}$ as follows:

$$\begin{aligned}
&\boldsymbol{B}_{\text{div}} : D(\boldsymbol{B}_{\text{div}}) \subset \mathcal{V}_{\text{div}} \to \mathcal{W}_{\text{div}}, \\
&D(\boldsymbol{B}_{\text{div}}) := \{\boldsymbol{u} \in \mathcal{V}_{\text{div}} \mid \boldsymbol{u} \in H_2(\Omega, \mathbb{R}^n),\ \boldsymbol{\Xi}\boldsymbol{\Sigma}(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{0} \text{ on } \Gamma\}, \\
&\boldsymbol{B}_{\text{div}}(\boldsymbol{u}) := \nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{u}) - \nabla q(\boldsymbol{u}), \\
&\boldsymbol{C}_{\text{div}} : D(\boldsymbol{C}_{\text{div}}) \subset \mathcal{W}_{\text{div}} \to \mathcal{V}_{\text{div}}, \\
&D(\boldsymbol{C}_{\text{div}}) := \{\boldsymbol{\xi} \in \mathcal{W}_{\text{div}} \mid \boldsymbol{\xi} \in H_1(\Omega, \mathbb{R}^n)\}, \\
&\boldsymbol{C}_{\text{div}}(\boldsymbol{\xi}) := \rho^{-1}\boldsymbol{\xi}, \\
&\boldsymbol{A}_{\text{div}} : D(\boldsymbol{A}_{\text{div}}) \subset \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}} \to \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}, \\
&D(\boldsymbol{A}_{\text{div}}) := D(\boldsymbol{B}_{\text{div}}) \times D(\boldsymbol{C}_{\text{div}}), \\
&\boldsymbol{A}_{\text{div}}(\boldsymbol{u}, \boldsymbol{\xi}) := (\boldsymbol{C}_{\text{div}}(\boldsymbol{\xi}), \boldsymbol{B}_{\text{div}}(\boldsymbol{u})),
\end{aligned} \tag{5.2}$$

where $q = q(\boldsymbol{u})$ is determined from

$$\begin{aligned} \nabla \cdot [\rho^{-1}\nabla q] &= \nabla \cdot [\rho^{-1}\nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{u})] \quad \text{in } \Omega, \\ q &= \boldsymbol{N} \cdot \boldsymbol{\Sigma}(\boldsymbol{u})\boldsymbol{N} \qquad\qquad \text{on } \Gamma. \end{aligned} \tag{5.3}$$

Remarks 5.1. 1. The boundary condition appearing in the definition of $D(\boldsymbol{B}_{\text{div}})$ is a homogeneous version of that in $D(\boldsymbol{B})$, and the system for $q(\boldsymbol{u})$ is a "homogeneous" version of the system for $\lambda(\boldsymbol{u})$.

2. For any $\phi \in \overset{\circ}{H}_1(\Omega, \mathbb{R})$ we have

$$\left\langle \boldsymbol{B}_{\text{div}}(\boldsymbol{u}), \rho^{-1}\nabla\phi \right\rangle_0 = 0. \tag{5.4}$$

This follows from (5.3).

3. For any $\boldsymbol{\xi} \in D(\boldsymbol{C}_{\text{div}})$ note that $\nabla \cdot \boldsymbol{C}_{\text{div}}(\boldsymbol{\xi}) = 0$.

A problem closely associated with (4.7) is the following:

Given $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$ and $(\boldsymbol{v}_0, \boldsymbol{w}_0) \in D(\boldsymbol{A}_{\text{div}})$, find a curve $(\boldsymbol{v}, \boldsymbol{w}) : [0, T] \to D(\boldsymbol{A}_{\text{div}}) \subset \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$ such that $(\boldsymbol{v}, \boldsymbol{w})(0) = (\boldsymbol{v}_0, \boldsymbol{w}_0)$ and

$$\begin{Bmatrix} \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{w}} \end{Bmatrix} = \boldsymbol{A}_{\text{div}} \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{Bmatrix} \qquad \forall t \in (0, T]. \tag{5.5}$$

The relation between (5.5) and (4.7) is established in the following proposition.

Proposition 5.1. *Let* $\boldsymbol{b} \in H_0(\Omega, \mathbb{R}^n)$, $\boldsymbol{h} \in H_{\frac{1}{2}}(\Gamma, \mathbb{R}^n)$ *and* $(\boldsymbol{u}_0, \boldsymbol{\xi}_0) \in D(\boldsymbol{A})$ *be given. Then* $(\boldsymbol{u}, \boldsymbol{\xi}) : [0, T] \to D(\boldsymbol{A}) \subset \mathcal{V} \times \mathcal{W}$ *is a curve satisfying* (4.7) *if and only if the curve* $(\boldsymbol{v}, \boldsymbol{w}) : [0, T] \to D(\boldsymbol{A}_{\text{div}}) \subset \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$*, defined by*

$$(\boldsymbol{v}(t), \boldsymbol{w}(t)) = (\boldsymbol{u}(t) - \boldsymbol{u}_0, \boldsymbol{\xi}(t) - \boldsymbol{\xi}_0), \tag{5.6}$$

satisfies (5.5) *with data*

$$\begin{aligned} \boldsymbol{f} &= \rho^{-1}[\boldsymbol{\xi}_0 - \nabla\mu(\boldsymbol{\xi}_0)], \\ \boldsymbol{g} &= \nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{u}_0) - \nabla\lambda(\boldsymbol{u}_0) + \boldsymbol{b}, \\ (\boldsymbol{v}_0, \boldsymbol{w}_0) &= (\boldsymbol{0}, \boldsymbol{0}). \end{aligned} \tag{5.7}$$

Proof. Assume $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ is a curve satisfying (4.7) with the given data. Then, for any $\phi \in \overset{\circ}{H}_1(\Omega, \mathbb{R})$ we have

$$\left.\begin{aligned} \frac{d}{dt}\langle \boldsymbol{u}, \nabla\phi\rangle_0 &= \langle \dot{\boldsymbol{u}}, \nabla\phi\rangle_0 = \langle \boldsymbol{C}(\boldsymbol{\xi}), \nabla\phi\rangle_0 = 0 \\ \frac{d}{dt}\left\langle \boldsymbol{\xi}, \rho^{-1}\nabla\phi \right\rangle_0 &= \left\langle \dot{\boldsymbol{\xi}}, \rho^{-1}\nabla\phi \right\rangle_0 = \left\langle \boldsymbol{B}(\boldsymbol{u}) + \boldsymbol{b}, \rho^{-1}\nabla\phi \right\rangle_0 = 0 \end{aligned}\right\} \quad \forall t \in (0, T] \tag{5.8}$$

which implies that $(\boldsymbol{v}, \boldsymbol{w})(t)$ is in $\mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$ for all $t \in [0, T]$. Next, note that by construction we have $\boldsymbol{v}(t) \in H_2(\Omega, \mathbb{R}^n)$ and $\boldsymbol{w}(t) \in H_1(\Omega, \mathbb{R}^n)$. Because $\boldsymbol{u}(t) \in D(\boldsymbol{B})$, it follows that

$$\begin{aligned} \Xi\,\boldsymbol{\Sigma}(\boldsymbol{v}(t))\boldsymbol{N} &= \Xi\,\boldsymbol{\Sigma}(\boldsymbol{u}(t))\boldsymbol{N} - \Xi\,\boldsymbol{\Sigma}(\boldsymbol{u}_0)\boldsymbol{N} \\ &= \Xi\boldsymbol{h} - \Xi\boldsymbol{h} = \boldsymbol{0}, \end{aligned} \tag{5.9}$$

which implies that $(\boldsymbol{v}, \boldsymbol{w})(t)$ is in $D(\boldsymbol{A}_{\text{div}})$ for all $t \in [0, T]$.

Since $(\boldsymbol{u}, \boldsymbol{\xi})(t)$ satisfies (4.7), and $(\boldsymbol{u}, \boldsymbol{\xi}) = (\boldsymbol{v} + \boldsymbol{u}_0, \boldsymbol{w} + \boldsymbol{\xi}_0)$, we find that $(\boldsymbol{v}, \boldsymbol{w})(t)$ satisfies

$$\begin{aligned} \dot{\boldsymbol{v}} &= \rho^{-1}[\boldsymbol{w} - \nabla\mu(\boldsymbol{w})] + \rho^{-1}[\boldsymbol{\xi}_0 - \nabla\mu(\boldsymbol{\xi}_0)], \\ \dot{\boldsymbol{w}} &= \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{v}) - \nabla\lambda(\boldsymbol{v} + \boldsymbol{u}_0) + \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{u}_0) + \boldsymbol{b}. \end{aligned} \tag{5.10}$$

Noting that $\mu(\boldsymbol{w}) = 0$, and that

$$\lambda(\boldsymbol{v} + \boldsymbol{u}_0) = q(\boldsymbol{v}) + \lambda(\boldsymbol{u}_0), \tag{5.11}$$

it follows that $(\boldsymbol{v}, \boldsymbol{w})(t)$ satisfies (5.5) with

$$\begin{aligned} \boldsymbol{f} &= \rho^{-1}[\boldsymbol{\xi}_0 - \nabla\mu(\boldsymbol{\xi}_0)] \in \mathcal{V}_{\text{div}}, \\ \boldsymbol{g} &= \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{u}_0) - \nabla\lambda(\boldsymbol{u}_0) + \boldsymbol{b} \in \mathcal{W}_{\text{div}}, \\ (\boldsymbol{v}_0, \boldsymbol{w}_0) &= (\boldsymbol{0}, \boldsymbol{0}). \end{aligned} \tag{5.12}$$

The converse follows similarly.

To establish existence and uniqueness results for problem (4.7) on $\mathcal{V} \times \mathcal{W}$, it thus suffices to consider the auxiliary problem (5.5) on the closed subspace $\mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$. Moreover, to establish results for (5.5) we need only consider the homogeneous version of it, i.e., $(\boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{0}, \boldsymbol{0})$. If we can show that the homogeneous problem generates a semigroup, then the existence of solutions for the inhomogeneous problem will follow from a variation of constants formula.

5.2. The Semigroup Approach

Consider the Hilbert space $\mathcal{X} = \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$ with inner-product defined by

$$\langle(\boldsymbol{u}, \boldsymbol{\xi}), (\boldsymbol{v}, \boldsymbol{w})\rangle_{\mathcal{X}} = \langle\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{V}} + \langle\boldsymbol{\xi}, \boldsymbol{w}\rangle_{\mathcal{W}}. \tag{5.13}$$

The goal of this section is to show that the operator $\boldsymbol{A}_{\text{div}} : D(\boldsymbol{A}_{\text{div}}) \subset \mathcal{X} \to \mathcal{X}$ defined in (5.5) is the infinitesimal generator of a (C^0) semigroup $\{\boldsymbol{S}(t) \mid t \geqq 0\}$ on $\mathcal{X}$. Recall that, if $\boldsymbol{S}(t)$ is the semigroup generated by $\boldsymbol{A}_{\text{div}}$, then for any $x_0 \in D(\boldsymbol{A}_{\text{div}})$ the curve $\boldsymbol{x}(t) = \boldsymbol{S}(t)\boldsymbol{x}_0$ lies in $D(\boldsymbol{A}_{\text{div}})$ and satisfies

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_{\text{div}}\boldsymbol{x} \quad \text{and} \quad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

To establish the existence and uniqueness of a semigroup for $\boldsymbol{A}_{\text{div}}$ we will employ the following fundamental result.

Theorem 5.2 (Lumer-Phillips**).** *Let $\boldsymbol{A}_{\text{div}}$ be a linear operator on a Hilbert space $\mathcal{X}$. Then $\boldsymbol{A}_{\text{div}}$ is the generator of a (quasi-contractive) semigroup $\boldsymbol{S}(t)$ if and only if*

(i) *$D(\boldsymbol{A}_{\text{div}})$ is dense in $\mathcal{X}$,*
(ii) *$\exists\beta \geqq 0$ such that $\langle \boldsymbol{x}, \boldsymbol{A}_{\text{div}}\,\boldsymbol{x}\rangle_{\mathcal{X}} \leqq \beta\,||\boldsymbol{x}||^2_{\mathcal{X}},\ \forall \boldsymbol{x} \in D(\boldsymbol{A}_{\text{div}})$,*
(iii) *$(\gamma\boldsymbol{I} - \boldsymbol{A}_{\text{div}}) : D(\boldsymbol{A}_{\text{div}}) \to \mathcal{X}$ is surjective for γ sufficiently large.*

The next proposition establishes the existence of a semigroup for $\boldsymbol{A}_{\text{div}}$.

Proposition 5.3. *Assume the domain Ω is open, bounded and of class C^2. Then the operator $\boldsymbol{A}_{\text{div}}$ in* (5.5) *is the generator of a semigroup $\boldsymbol{S}(t)$ on $\mathcal{X}$.*

Proof. The result follows by direct verification of the conditions in the Lumer-Phillips Theorem.

1. To establish the inequality in (ii) let $(\boldsymbol{v}, \boldsymbol{w}) \in D(\boldsymbol{A}_{\text{div}})$ be arbitrary. Then

$$\begin{aligned}
&\langle(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{A}_{\text{div}}(\boldsymbol{v}, \boldsymbol{w})\rangle_{\mathcal{X}}\\
&\quad = \langle(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{C}_{\text{div}}(\boldsymbol{w}), \boldsymbol{B}_{\text{div}}(\boldsymbol{v}))\rangle_{\mathcal{X}}\\
&\quad = \langle \boldsymbol{v}, \boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_{\mathcal{V}} + \langle \boldsymbol{w}, \boldsymbol{B}_{\text{div}}(\boldsymbol{v})\rangle_{\mathcal{W}}\\
&\quad = \alpha\langle \boldsymbol{v}, \rho\boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0 + \langle \nabla\boldsymbol{C}_{\text{div}}(\boldsymbol{w}), \boldsymbol{\Sigma}(\boldsymbol{v})\rangle_0 + \left\langle \rho^{-1}\boldsymbol{w}, \boldsymbol{B}_{\text{div}}(\boldsymbol{v})\right\rangle_0.
\end{aligned}$$

Integrating the second term by parts, using the boundary condition on $\boldsymbol{\Sigma}(\boldsymbol{v})$ and using the definition of $q(\boldsymbol{v})$ gives

$$\begin{aligned}
\langle(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{A}_{\text{div}}(\boldsymbol{v}, \boldsymbol{w})\rangle_{\mathcal{X}} &= \alpha\langle \boldsymbol{v}, \rho\boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0 + \langle q(\boldsymbol{v})\boldsymbol{C}_{\text{div}}(\boldsymbol{w}), \boldsymbol{N}\rangle_{0,\Gamma}\\
&\quad - \langle \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{v}), \boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0 + \left\langle \rho^{-1}\boldsymbol{w}, \boldsymbol{B}_{\text{div}}(\boldsymbol{v})\right\rangle_0.
\end{aligned}$$

Applying the divergence theorem to the second term and using the fact that $\nabla\cdot\boldsymbol{C}_{\text{div}}(\boldsymbol{w}) = 0$ yields

$$\begin{aligned}
\langle(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{A}_{\text{div}}(\boldsymbol{v}, \boldsymbol{w})\rangle_{\mathcal{X}} &= \alpha\langle \boldsymbol{v}, \rho\boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0 + \langle \nabla q(\boldsymbol{v}), \boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0\\
&\quad - \langle \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{v}), \boldsymbol{C}_{\text{div}}(\boldsymbol{w})\rangle_0 + \left\langle \rho^{-1}\boldsymbol{w}, \boldsymbol{B}_{\text{div}}(\boldsymbol{v})\right\rangle_0\\
&= \alpha\langle \boldsymbol{v}, \boldsymbol{w}\rangle_0,
\end{aligned}$$

where the last line follows from the definitions of $\boldsymbol{B}_{\text{div}}$ and $\boldsymbol{C}_{\text{div}}$. Applying the Cauchy-Schwartz inequality we find that

$$\langle(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{A}_{\text{div}}(\boldsymbol{v}, \boldsymbol{w})\rangle_{\mathcal{X}} \leqq \frac{\alpha}{2}(||\rho^{1/2}\boldsymbol{v}||_0^2 + ||\rho^{-1/2}\boldsymbol{w}||_0^2) \leqq \beta\,||(\boldsymbol{v}, \boldsymbol{w})||^2_{\mathcal{X}},$$

where $\beta = \max(1, \alpha)/2$.

2. To establish condition (iii), for any $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{X}$ we consider the equation

$$(\gamma\boldsymbol{I} - \boldsymbol{A}_{\text{div}})(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{g}). \tag{5.14}$$

For γ sufficiently large, our goal is to show that given $(\boldsymbol{f}, \boldsymbol{g})$ there is a unique $(\boldsymbol{v}, \boldsymbol{w}) \in D(\boldsymbol{A}_{\text{div}})$ satisfying (5.14). Writing the above system in components and rearranging terms yields the system

$$\begin{aligned} \boldsymbol{w} &= \rho(\gamma \boldsymbol{v} - \boldsymbol{f}), \\ \gamma^2 \rho \boldsymbol{v} - \boldsymbol{B}_{\text{div}}(\boldsymbol{v}) &= \gamma \rho \boldsymbol{f} + \boldsymbol{g}. \end{aligned} \tag{5.15}$$

Hence, given $\boldsymbol{a} = \gamma \rho \boldsymbol{f} + \boldsymbol{g} \in \mathcal{W}_{\text{div}}$ we seek $\boldsymbol{v} \in D(\boldsymbol{B}_{\text{div}}) \subset \mathcal{V}_{\text{div}}$ such that

$$\boldsymbol{L}_{\gamma^2}(\boldsymbol{v}) := \gamma^2 \rho \boldsymbol{v} - \boldsymbol{B}_{\text{div}}(\boldsymbol{v}) = \boldsymbol{a}. \tag{5.16}$$

To solve (5.16) we consider an associated weak formulation obtained by multiplying by an arbitrary element $\boldsymbol{\eta}$ of $\mathcal{V}_{\text{div}}$ and integrating by parts. The resulting weak equation is

$$\gamma^2 \langle \rho \boldsymbol{v}, \boldsymbol{\eta} \rangle_0 + \langle \mathsf{C} \nabla \boldsymbol{v}, \nabla \boldsymbol{\eta} \rangle_0 = \langle \boldsymbol{a}, \boldsymbol{\eta} \rangle_0 \quad \forall\, \boldsymbol{\eta} \in \mathcal{V}_{\text{div}}. \tag{5.17}$$

For $\gamma^2 \geqq \alpha$ we note that the left-hand side is equivalent to the inner-product on $\mathcal{V}_{\text{div}} \subset H_1(\Omega, \mathbb{R}^n)$. Hence, by the Lax-Milgram Theorem [27], there exists a unique $\boldsymbol{v} \in \mathcal{V}_{\text{div}}$ satisfying (5.17). Using regularity results for Neumann boundary value problems for Stokes-type systems given in Giaquinta & Modica [19, Theorem 1.2, p. 198; Remark 2.6, p. 206] (slightly generalized to include the term $\gamma^2 \langle \rho \boldsymbol{v}, \boldsymbol{\eta} \rangle_0$ above), we have $\boldsymbol{v} \in \mathcal{V}_{\text{div}} \cap H_2(\Omega, \mathbb{R}^n)$.

It thus remains to show that $\boldsymbol{v} \in D(\boldsymbol{B}_{\text{div}})$, that is, to show $\boldsymbol{v}$ satisfies the appropriate boundary conditions. To this end, consider the linear functional $\ell : \mathring{H}_1 \to \mathbb{R}$ defined for any $\boldsymbol{\eta} \in \mathring{H}_1$ by

$$\ell(\boldsymbol{\eta}) = \gamma^2 \langle \rho \boldsymbol{v}, \boldsymbol{\eta} \rangle_0 + \langle \mathsf{C} \nabla \boldsymbol{v}, \nabla \boldsymbol{\eta} \rangle_0 - \langle \boldsymbol{a}, \boldsymbol{\eta} \rangle_0. \tag{5.18}$$

It is straightforward to show that there is a constant C, independent of $\boldsymbol{\eta}$, such that

$$|\ell(\boldsymbol{\eta})| \leqq C ||\boldsymbol{\eta}||_1.$$

Thus $\ell \in H_{-1} = (\mathring{H}_1)'$. From (5.17) we have

$$\ell(\boldsymbol{\eta}) = 0 \quad \forall\, \boldsymbol{\eta} \in \mathring{H}_1 \cap \mathcal{V}_{\text{div}}$$

and by Propositions 1.1 and 1.2 in Temam [42, p. 14] there exists a $\hat{q} \in H_0$ such that $\langle \hat{q}, 1 \rangle_0 = 0$ and

$$\ell(\boldsymbol{\eta}) = \langle \hat{q}, \nabla \cdot \boldsymbol{\eta} \rangle_0 \quad \forall\, \boldsymbol{\eta} \in \mathring{H}_1,$$

or

$$\gamma^2 \langle \rho \boldsymbol{v}, \boldsymbol{\eta} \rangle_0 + \langle \boldsymbol{\Sigma}(\boldsymbol{v}), \nabla \boldsymbol{\eta} \rangle_0 - \langle \boldsymbol{a}, \boldsymbol{\eta} \rangle_0 = \langle \hat{q}, \nabla \cdot \boldsymbol{\eta} \rangle_0 \quad \forall\, \boldsymbol{\eta} \in \mathring{H}_1. \tag{5.19}$$

Using (5.19) together with the fact that $\boldsymbol{v}$ is in H_2 it is straightforward to show that $\hat{q} \in H_1$. Moreover, integrating by parts in (5.19) we find that

$$\nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{v}) - \nabla \hat{q} = \gamma^2 \rho \boldsymbol{v} - \boldsymbol{a} \quad \text{in } \Omega. \tag{5.20}$$

Now, consider a mapping $\mathsf{P} : H_1 \to \overset{\circ}{H}_1$ defined by

$$\mathsf{P}(\boldsymbol{\eta}) = \boldsymbol{\theta}(\boldsymbol{\eta}) - \boldsymbol{\Theta}(\boldsymbol{\eta}),$$

where $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\eta})$ is defined by the Laplace equation

$$\begin{aligned} \Delta\boldsymbol{\theta} &= \mathbf{0} && \text{in } \Omega, \\ \boldsymbol{\theta} &= \boldsymbol{\eta} - \left[\frac{\int_\Gamma \boldsymbol{\eta}\cdot\boldsymbol{N}\, d\Gamma}{\int_\Gamma d\Gamma}\right]\boldsymbol{N} && \text{in } \Gamma, \end{aligned}$$

and $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\boldsymbol{\eta})$ is defined by the Stokes equations

$$\begin{aligned} \Delta\boldsymbol{\Theta} - \nabla\xi &= \mathbf{0} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\Theta} &= 0 && \text{in } \Omega, \\ \boldsymbol{\Theta} &= \boldsymbol{\theta} && \text{in } \Gamma. \end{aligned}$$

That $\boldsymbol{\theta}(\boldsymbol{\eta})$ is well defined follows from the standard theory for elliptic equations, see, e.g., [30], and that $\boldsymbol{\Theta}(\boldsymbol{\eta})$ is well defined follows from the standard theory for the Stokes equations, see, e.g., [42]. Using the definition of P together with (5.19) we thus have

$$\begin{aligned} \gamma^2\langle\rho\boldsymbol{v}, \mathsf{P}(\boldsymbol{\eta})\rangle_0 + \langle\boldsymbol{\Sigma}(\boldsymbol{v}), \nabla\mathsf{P}(\boldsymbol{\eta})\rangle_0 - \langle\boldsymbol{a}, \mathsf{P}(\boldsymbol{\eta})\rangle_0 \\ = \left\langle\hat{q}, \nabla\cdot\mathsf{P}(\boldsymbol{\eta})\right\rangle_0 \quad \forall\,\boldsymbol{\eta}\in H_1, \end{aligned} \tag{5.21}$$

and using the fact that $\boldsymbol{\Theta}(\boldsymbol{\eta}) \in \mathcal{V}_{\text{div}}$ together with (5.17) the above equation reduces to

$$\begin{aligned} \gamma^2\langle\rho\boldsymbol{v}, \boldsymbol{\theta}(\boldsymbol{\eta})\rangle_0 + \langle\boldsymbol{\Sigma}(\boldsymbol{v}), \nabla\boldsymbol{\theta}(\boldsymbol{\eta})\rangle_0 - \langle\boldsymbol{a}, \boldsymbol{\theta}(\boldsymbol{\eta})\rangle_0 \\ = \left\langle\hat{q}, \nabla\cdot\boldsymbol{\theta}(\boldsymbol{\eta})\right\rangle_0, \quad \forall\,\boldsymbol{\eta}\in H_1. \end{aligned} \tag{5.22}$$

Integrating the above equation by parts and using (5.20) we obtain

$$\begin{aligned} 0 &= \left\langle\boldsymbol{\theta}(\boldsymbol{\eta}), \boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N} - \hat{q}\boldsymbol{N}\right\rangle_{0,\Gamma} \\ &= \left\langle\boldsymbol{\eta}, \boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N} - \hat{q}\boldsymbol{N}\right\rangle_{0,\Gamma} \\ &\quad - \left[\frac{\int_\Gamma \boldsymbol{\eta}\cdot\boldsymbol{N}\, d\Gamma}{\int_\Gamma d\Gamma}\right]\left\langle\boldsymbol{N}, \boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N} - \hat{q}\boldsymbol{N}\right\rangle_{0,\Gamma} \qquad \forall\,\boldsymbol{\eta}\in H_1. \end{aligned} \tag{5.23}$$

To establish that $\boldsymbol{v}$ satisfies the appropriate boundary condition consider any $\boldsymbol{\eta}\in H_1$ such that $\boldsymbol{\eta} = \boldsymbol{\Xi}\,\boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N}$ on Γ where $\boldsymbol{\Xi}(\boldsymbol{X})$ is the projection onto the tangent plane to Γ at $\boldsymbol{X}$. For any such $\boldsymbol{\eta}$ we have $\boldsymbol{\eta}\cdot\boldsymbol{N} = 0$ on Γ and (5.23) yields

$$\langle\boldsymbol{\Xi}\,\boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N}, \boldsymbol{\Xi}\,\boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N}\rangle_{0,\Gamma} = 0. \tag{5.24}$$

The above equation implies $\boldsymbol{\Xi}\,\boldsymbol{\Sigma}(\boldsymbol{v})\boldsymbol{N} = \mathbf{0}$ on Γ which is the appropriate boundary condition for $\boldsymbol{v}$. We thus have $\boldsymbol{v}\in D(\boldsymbol{B}_{\text{div}})$ and $(5.15)_1$ yields $\boldsymbol{w}\in D(\boldsymbol{C}_{\text{div}})$. The surjectivity condition is thus proved.

3. Since $D(\boldsymbol{C}_{\text{div}})$ is dense in $\mathcal{W}_{\text{div}}$, condition (i) will follow from the denseness of $D(\boldsymbol{B}_{\text{div}})$ in $\mathcal{V}_{\text{div}}$. To establish the denseness of $D(\boldsymbol{B}_{\text{div}})$ consider any $\boldsymbol{v} \in [D(\boldsymbol{B}_{\text{div}})]^{\perp}$. Then, for any $\boldsymbol{u} \in D(\boldsymbol{B}_{\text{div}})$, we have

$$\begin{aligned} 0 = \langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\mathcal{V}} &= \alpha \langle \boldsymbol{v}, \rho \boldsymbol{u} \rangle_0 + \langle \nabla \boldsymbol{v}, \boldsymbol{\Sigma}(\boldsymbol{u}) \rangle_0 \\ &= \langle \alpha \rho \boldsymbol{u} - \nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{u}) + \nabla q(\boldsymbol{u}), \boldsymbol{v} \rangle_0 \\ &= \langle \boldsymbol{L}_\alpha(\boldsymbol{u}), \boldsymbol{v} \rangle_0 = \langle \boldsymbol{L}_\alpha(\boldsymbol{u}), \rho \boldsymbol{v} \rangle_{\mathcal{W}}. \end{aligned}$$

Since the operator $\boldsymbol{L}_\alpha$ maps $D(\boldsymbol{B}_{\text{div}})$ onto $\mathcal{W}_{\text{div}}$ and $\rho \boldsymbol{v} \in \mathcal{W}_{\text{div}}$, we deduce that $\boldsymbol{v} = \boldsymbol{0}$, thus proving the denseness of $D(\boldsymbol{B}_{\text{div}})$.

5.3. Proof of Theorem 4.1

By Proposition 5.3 the operator $\boldsymbol{A}_{\text{div}}$ generates a semigroup $\boldsymbol{S}_{\text{div}}(t)$ on $\mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$. Given any $(\boldsymbol{v}_0, \boldsymbol{w}_0) \in D(\boldsymbol{A}_{\text{div}})$ and any $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$, we can solve (5.5) via the variation of constants formula

$$(\boldsymbol{v}, \boldsymbol{w})(t) = \boldsymbol{S}_{\text{div}}(t)(\boldsymbol{v}_0, \boldsymbol{w}_0) + \int_0^t \boldsymbol{S}_{\text{div}}(t-\tau)(\boldsymbol{f}, \boldsymbol{g})\, d\tau. \tag{5.25}$$

In particular, $(\boldsymbol{v}, \boldsymbol{w})(t)$ is a continuously differentiable curve in $D(\boldsymbol{A}_{\text{div}}) \subset \mathcal{V}_{\text{div}} \times \mathcal{W}_{\text{div}}$ satisfying (5.5) and $(\boldsymbol{v}, \boldsymbol{w})(0) = (\boldsymbol{v}_0, \boldsymbol{w}_0)$, see, e.g., [24, Chapter IX, Section 1.5]. Theorem 4.1 now follows from Proposition 5.1 by taking

$$\begin{aligned} \boldsymbol{f} &= \rho^{-1}[\boldsymbol{\xi}_0 - \nabla \mu(\boldsymbol{\xi}_0)] \in \mathcal{V}_{\text{div}}, \\ \boldsymbol{g} &= \nabla \cdot \boldsymbol{\Sigma}(\boldsymbol{u}_0) - \nabla \lambda(\boldsymbol{u}_0) + \boldsymbol{b} \in \mathcal{W}_{\text{div}}, \\ (\boldsymbol{v}_0, \boldsymbol{w}_0) &= (\boldsymbol{0}, \boldsymbol{0}). \end{aligned} \tag{5.26}$$

6. Discussion

We have shown that when considering finite- and infinite-dimensional Lagrangian dynamical systems subject to holonomic constraints, there can be some advantage in enforcing velocity-level constraints either instead of, or in addition to, configuration-level constraints. All the formulations we consider are *ambient-space* formulations: they are defined (at least formally) on the whole of an ambient space and they possess a physical solution set, that is, an invariant set on which physical balance laws and constraints are satisfied. Depending on how constraints are introduced into the underlying action principle, using multipliers of striction or pressure type, different ambient-space formulations can be constructed for which the physical solution set is either unstable, the level set of a first integral, or exponentially attractive in an appropriate sense.

6.1. Ambient-Space Formulations

As mentioned in the Introduction, our motivations for studying ambient-space formulations are twofold. First, we believe that these types of formulations may be useful within the context of numerical simulation and analysis. In particular, the ability to control the stability properties of the physical solution set while maintaining a Hamiltonian structure is a potentially important freedom that does not appear to have been fully exploited in the literature on constrained systems, especially in the infinite-dimensional case. Second, ambient-space formulations can sometimes allow standard analysis techniques, such as stability calculations via Lyapunov arguments [11] or existence and regularity theory as considered here, to be brought to bear more simply and concretely than for analogous formulations restricted to constraint manifolds.

We believe ambient-space formulations to be of practical interest for both finite- and infinite-dimensional systems. Many constrained Lagrangian systems of contemporary interest are intrinsically finite-dimensional, and ambient-space formulations could provide a practical means for their numerical treatment. Indeed, various studies have been made along these lines [4, 28]. However, the Hamiltonian ambient-space formulation introduced here does not seem to have appeared before, and perhaps deserves further study in such contexts.

In this article we have considered ambient-space formulations for both finite- and infinite-dimensional constrained Lagrangian systems. However, we did not probe into the connections between a given infinite-dimensional system and associated finite-dimensional approximations, such as would arise from spatial discretization. If, for example, one considers spatial discretizations that preserve Lagrangian structure, then one can immediately consider two finite-dimensional ambient-space approximations depending on whether one first passes to an ambient-space formulation and then discretizes, or vice-versa. In such cases the numerical treatment of constrained infinite-dimensional systems via an ambient-space formulation would seem to necessitate a firm understanding of how the discretization process interacts with the passage to an ambient space.

6.2. Multiple Multipliers

When a linear combination of the original configuration-level constraint and its associated velocity-level constraint, as in (2.9) or (3.20), are introduced into the action principle, we refer to the associated multiplier as being of striction type. This type of multiplier can be eliminated via an appropriate minimization, and the resulting ambient-space formulation is Hamiltonian. Moreover, the formulation possesses an exponentially attractive physical solution set when the parameter α appearing in (2.9) or (3.20) is positive. This observation, which appears to be new in both the finite- and infinite-dimensional cases, is one extension of the impetus-striction approach as described in [31, 11], which constructed only the neutrally stable case.

For infinite-dimensional problems, we showed that use of a single multiplier field of striction type can lead to boundary evolution equations that may be unusual to use or analyze. However, in our second extension of the impetus-striction method, we showed that boundary evolution equations can be avoided by introducing multiplier fields of both striction and pressure type. The physical solution set in the resulting multi-multiplier formulation, which no longer need be Hamiltonian, enjoyed all the stability properties of the pure striction approach. For the specific initial-boundary value problem in linearized incompressible elastodynamics corresponding to a body with a specified traction on its boundary, the multi-multiplier formulation was proven to be well posed in the sense stated in Theorem 4.1.

6.3. Impetus and Impulse

When constraints of the form (2.9) or (3.20) are introduced into the action principle the natural conjugate variable is not the classic momentum, but a related quantity that we call the *impetus* [31, 11]. In each case, impetus determines the striction and velocity fields according to a rule of decomposition guaranteeing that the constraints remain satisfied. The impetus and striction variables have an interesting physical meaning that can be explained as follows. Consider first an unconstrained finite-dimensional system, analogous to those considered in Section 2, subject to an additional external force $\boldsymbol{f}(t)$ so that

$$\dot{\boldsymbol{q}} = \quad m^{-1}\boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = -DV(\boldsymbol{q}) + \boldsymbol{f}. \tag{6.1}$$

Suppose the system at time $t = 0$ is in a prescribed configuration $\boldsymbol{q}_0$ with prescribed momentum $\boldsymbol{p}_0$. Assuming the force $\boldsymbol{f}(t)$ to be of short duration $\varepsilon > 0$, we integrate (6.1) to obtain

$$[\![\boldsymbol{q}]\!]_0^\varepsilon = \mathcal{O}(\varepsilon), \qquad [\![\boldsymbol{p}]\!]_0^\varepsilon = \int_0^\varepsilon \boldsymbol{f}(t)\,dt + \mathcal{O}(\varepsilon), \tag{6.2}$$

where $[\![\boldsymbol{q}]\!]_0^\varepsilon = \boldsymbol{q}_\varepsilon - \boldsymbol{q}_0$, and so on.

If $\boldsymbol{f}(t)$ is a classic *impulsive force*, that is, a force of such large magnitude and short duration that it can be well approximated by the Dirac distribution $\boldsymbol{i}\delta(t)$ with

$$\int_0^\varepsilon \boldsymbol{f}(t)\,dt = \boldsymbol{i} \quad \forall\,\varepsilon > 0, \tag{6.3}$$

then we call $\boldsymbol{i}$ the *impulse* associated with $\boldsymbol{f}(t)$, and (6.2) yields the jump conditions

$$[\![\boldsymbol{q}]\!]_0 = \boldsymbol{0}, \qquad [\![\boldsymbol{p}]\!]_0 = \boldsymbol{i}, \tag{6.4}$$

where $[\![\boldsymbol{q}]\!]_0 = \lim_{\varepsilon\downarrow 0} [\![\boldsymbol{q}]\!]_0^\varepsilon$, etc. Thus, if impulse is defined as the time integral of an impulsive force, then the impulse is the jump in momentum. In particular, if the system is at rest at time $t = 0$, the impulse is the momentum at time $t = 0^+$.

Now consider the (holonomically) constrained finite-dimensional system described by equations (2.4) in Section 2. When the system is subject to an additional external force $\boldsymbol{f}(t)$ the equations become

$$\begin{aligned} \dot{\boldsymbol{q}} &= m^{-1}\boldsymbol{p}, \\ \dot{\boldsymbol{p}} &= -DV(\boldsymbol{q}) - \lambda Dg(\boldsymbol{q}) + \boldsymbol{f}, \\ 0 &= g(\boldsymbol{q}). \end{aligned} \tag{6.5}$$

Suppose the system at time $t = 0$ has a prescribed configuration $\boldsymbol{q}_0$ satisfying $g(\boldsymbol{q}_0) = 0$, and has a prescribed momentum $\boldsymbol{p}_0$ consistent with $(6.5)_3$ in the sense that $Dg(\boldsymbol{q}_0) \cdot m^{-1}\boldsymbol{p}_0 = 0$. Integrating as before we find

$$\begin{aligned} [\![\boldsymbol{q}]\!]_0^\varepsilon &= \mathcal{O}(\varepsilon), \\ [\![\boldsymbol{p}]\!]_0^\varepsilon &= -\left[\int_0^\varepsilon \lambda(t)\,dt\right] Dg(\boldsymbol{q}_0) + \int_0^\varepsilon \boldsymbol{f}(t)\,dt + \mathcal{O}(\varepsilon). \end{aligned} \tag{6.6}$$

If $\boldsymbol{f}(t)$ is an impulsive force as before with impulse $\boldsymbol{i} \neq \boldsymbol{0}$, then in order that the momentum $\boldsymbol{p}_\varepsilon$ satisfy $Dg(\boldsymbol{q}_\varepsilon) \cdot m^{-1}\boldsymbol{p}_\varepsilon = 0$ for all $\varepsilon > 0$, the multiplier $\lambda(t)$ must admit the limit

$$\int_0^\varepsilon \lambda(t)\,dt \to \frac{Dg(\boldsymbol{q}_0) \cdot m^{-1}\boldsymbol{i}}{Dg(\boldsymbol{q}_0) \cdot m^{-1}Dg(\boldsymbol{q}_0)} \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{6.7}$$

Thus the pressure-like multiplier is an impulsive variable.

On the other hand, in view of Proposition 2.3, the striction variable $\mu(t)$ for (6.5) satisfies the equation

$$\dot{\mu} - \alpha\mu = -\lambda, \tag{6.8}$$

where $\alpha \in \mathbb{R}$ is a parameter. Integrating, we obtain

$$[\![\mu]\!]_0^\varepsilon = -\int_0^\varepsilon \lambda(t)\,dt + \mathcal{O}(\varepsilon), \tag{6.9}$$

so that in the limit $\varepsilon \downarrow 0$ the striction multiplier has a simple discontinuity according to the relation

$$[\![\mu]\!]_0 = -\frac{Dg(\boldsymbol{q}_0) \cdot m^{-1}\boldsymbol{i}}{Dg(\boldsymbol{q}_0) \cdot m^{-1}Dg(\boldsymbol{q}_0)}. \tag{6.10}$$

Moreover, from $(6.6)_2$ we find the jump in momentum to be

$$[\![\boldsymbol{p}]\!]_0 = \boldsymbol{i} - \left[\frac{Dg(\boldsymbol{q}_0) \cdot m^{-1}\boldsymbol{i}}{Dg(\boldsymbol{q}_0) \cdot m^{-1}Dg(\boldsymbol{q}_0)}\right] Dg(\boldsymbol{q}_0). \tag{6.11}$$

Finally, the impetus variable $\boldsymbol{\xi}$ for (6.5) by definition satisfies

$$\boldsymbol{\xi}(t) = m\dot{\boldsymbol{q}}(t) - \mu(t)Dg(\boldsymbol{q}(t)), \tag{6.12}$$

so that

$$[\![\boldsymbol{\xi}]\!]_0^\varepsilon = [\![\boldsymbol{p}]\!]_0^\varepsilon - [\![\mu]\!]_0^\varepsilon Dg(\boldsymbol{q}_0) + \mathcal{O}(\varepsilon). \tag{6.13}$$

Combining (6.13), (6.9) and (6.6), and passing to the limit $\varepsilon \downarrow 0$, leads to the jump conditions

$$[\![\boldsymbol{q}]\!]_0 = \boldsymbol{0}, \qquad [\![\boldsymbol{\xi}]\!]_0 = \boldsymbol{i}. \tag{6.14}$$

Thus, for a constrained system, *the impulse is equal to the jump in impetus*. In particular, if the system has zero impetus at time $t = 0$, the impulse is the impetus at time $t = 0^+$. Furthermore, from (6.10) and (6.11) we see that, for a given impulse, the jump in momentum and the jump in striction are determined according to a decomposition rule exactly analogous to (6.12), namely

$$\boldsymbol{i} = [\![\boldsymbol{p}]\!]_0 - [\![\mu]\!]_0 Dg(\boldsymbol{q}_0). \tag{6.15}$$

We now consider the case of linearized incompressible elasticity discussed in Section 3. Consider the system described by (3.3) and suppose this system is subject to a body force field $\boldsymbol{b}$ in Ω and a boundary pressure field h on Γ_σ, so the equations under consideration are

$$\begin{aligned}
\rho\,\ddot{\boldsymbol{u}} &= \nabla\cdot\boldsymbol{\Sigma} - \nabla p + \boldsymbol{b} && \text{in } \Omega\times(0,T],\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega\times[0,T],\\
\boldsymbol{\Sigma}\boldsymbol{N} - p\boldsymbol{N} &= -h\boldsymbol{N} && \text{in } \Gamma_\sigma\times[0,T],\\
\boldsymbol{\Xi}\,\boldsymbol{\Sigma}\boldsymbol{N} &= \boldsymbol{0} && \text{in } \Gamma_s\times[0,T],\\
\boldsymbol{u}\cdot\boldsymbol{N} &= 0 && \text{in } \Gamma_s\times[0,T],\\
\boldsymbol{u} &= \boldsymbol{0} && \text{in } \Gamma_u\times[0,T].
\end{aligned} \tag{6.16}$$

Suppose the system at time $t = 0$ has a prescribed configuration $\boldsymbol{u}_0$ satisfying the boundary conditions and the constraint $\nabla\cdot\boldsymbol{u}_0 = 0$, and has a prescribed velocity $\dot{\boldsymbol{u}}_0$ consistent with $(6.16)_2$ in the sense that $\nabla\cdot\dot{\boldsymbol{u}}_0 = 0$. Assuming that the external loads $\boldsymbol{b}$ and h are of short duration $\varepsilon > 0$, and that (6.16) admits sufficiently regular solutions for these loads, we formally integrate $(6.16)_1$ to obtain

$$[\![\rho\dot{\boldsymbol{u}}]\!]_0^\varepsilon = -\nabla\left[\int_0^\varepsilon p\,dt\right] + \int_0^\varepsilon \boldsymbol{b}\,dt + \mathcal{O}(\varepsilon) \quad \text{in } \Omega, \tag{6.17}$$

and integrate $(6.16)_3$ to obtain

$$\int_0^\varepsilon p\,dt = \int_0^\varepsilon h\,dt + \mathcal{O}(\varepsilon) \quad \text{on } \Gamma_\sigma. \tag{6.18}$$

If the body force $\boldsymbol{b}$ and boundary pressure h are impulsive loads with impulses $\boldsymbol{i}_\Omega$ and i_{Γ_σ} respectively, so that

$$\int_0^\varepsilon \boldsymbol{b}\,dt = \boldsymbol{i}_\Omega \quad \text{and} \quad \int_0^\varepsilon h\,dt = i_{\Gamma_\sigma} \quad \forall\,\varepsilon > 0,$$

then in order that the velocity satisfy $\nabla\cdot\dot{\boldsymbol{u}}_\varepsilon = 0$ for all $\varepsilon > 0$, the pressure multiplier p must admit the formal limit

$$\int_0^\varepsilon p\,dt \to \phi$$

for an appropriate field ϕ depending on $\boldsymbol{i}_\Omega$ and i_{Γ_σ}. The pressure multiplier is thus an impulsive field. In view of (6.17), the field ϕ satisfies the equation

$$\nabla\cdot(\rho^{-1}\nabla\phi) = \nabla\cdot(\rho^{-1}\boldsymbol{i}_\Omega) \quad \text{in } \Omega$$

and, in view of (6.18), the boundary condition

$$\phi = i_{\Gamma_\sigma} \quad \text{on } \Gamma_\sigma.$$

Assuming the normal component of the velocity $\dot{\boldsymbol{u}}_\varepsilon$ has a well defined trace on $\Gamma_u \cup \Gamma_s$, we have also the boundary condition

$$\nabla\phi \cdot \boldsymbol{N} = \boldsymbol{i}_\Omega \cdot \boldsymbol{N} \quad \text{on } \Gamma_u \cup \Gamma_s$$

obtained from (6.17) together with $(6.16)_{5,6}$.

On the other hand, in view of Proposition 3.3, the striction variable μ for (6.16) satisfies the equation

$$\dot{\mu} - \alpha\mu = p, \tag{6.19}$$

where $\alpha \in \mathbb{R}$ is a parameter. Integrating we obtain

$$[\![\mu]\!]_0^\varepsilon = \int_0^\varepsilon p(t)\, dt + \mathcal{O}(\varepsilon), \tag{6.20}$$

and we find that, in the limit $\varepsilon \downarrow 0$, the striction multiplier has a discontinuity in time according to

$$[\![\mu]\!]_0 = \phi,$$

where ϕ is as defined above. Moreover, from (6.17) we find the jump in physical momentum to be

$$[\![\rho\dot{\boldsymbol{u}}]\!]_0 = -\nabla[\![\mu]\!]_0 + \boldsymbol{i}_\Omega. \tag{6.21}$$

Finally, the impetus variable $\boldsymbol{\xi} = (\boldsymbol{\zeta}, \nu)$ for (6.16) by definition satisfies

$$\begin{aligned} \boldsymbol{\zeta}(t) &= \rho\dot{\boldsymbol{u}}(t) + \nabla\mu(t) && \text{in } \Omega, \\ \nu(t) &= \mu(t) && \text{on } \Gamma_\sigma, \end{aligned} \tag{6.22}$$

and thus

$$\begin{aligned} [\![\boldsymbol{\zeta}]\!]_0^\varepsilon &= [\![\rho\dot{\boldsymbol{u}}]\!]_0^\varepsilon + \nabla[\![\mu]\!]_0^\varepsilon && \text{in } \Omega, \\ [\![\nu]\!]_0^\varepsilon &= [\![\mu]\!]_0^\varepsilon && \text{on } \Gamma_\sigma. \end{aligned} \tag{6.23}$$

Combining (6.23), (6.20), (6.18) and (6.17), and passing to the limit $\varepsilon \downarrow 0$, leads to the jump conditions

$$\begin{aligned} [\![\boldsymbol{\zeta}]\!]_0 &= \boldsymbol{i}_\Omega && \text{in } \Omega, \\ [\![\nu]\!]_0 &= i_{\Gamma_\sigma} && \text{on } \Gamma_\sigma. \end{aligned} \tag{6.24}$$

Thus, for a constrained infinite-dimensional system such as linearized incompressible elasticity, the body impulse equals the jump in body impetus, and the boundary impulse equals the jump in boundary impetus.

The above arguments are in concordance with the classic notions of impulsive force and impulse dating back to KELVIN & LAMB [26]. They show that, just as momentum and impulse are related but different quantities for unconstrained systems, impetus and impulse are related but different quantities for constrained systems. In the particular context of fluid dynamics, variables analogous to what we call impetus have been called “impulse variables” [41, 37, 9], and in one instance a “magnetization variable” [8]. We prefer to call our conjugate variable impetus, rather than impulse, for the above reasons and the fact that the name impetus signals unequivocally that a constrained system is at hand. Moreover, use of the term impulse can lead to a second confusion because, while the classic definition is as the time integral of an applied impulsive force, some authors reserve the term for the conserved quantity associated with the symmetry of homogeneous media (see, e.g., [5] for a general discussion, or [11] for an analysis in which impetus, momentum and impulse in the sense of [5] all play important and distinct roles).

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