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THEOREMS ON THE STOKESIAN HYDRODYNAMICS OF A RIGID FILAMENT IN THE LIMIT OF VANISHING RADIUS*

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Abstract. The transport dynamics of a rigid filament in a slow viscous flow modeled by the Stokes equations in a three-dimensional domain are considered. The relation between the transport velocities, external loads, and far-field flow around the filament is studied in the singular limit as the radius of the filament tends to zero, and the filament collapses to a curve, which may be an open arc or closed loop. Beginning from an initially implicit relation defined through the solution of a boundary-value problem, a potential theoretic representation is used to show that the relation between the transport velocities, external loads, and far-field flow remains well defined and remarkably becomes explicit in the limit of vanishing radius. Theorems establishing the form of this relation in different cases are stated, proved, and illustrated with examples. Special considerations at the ends of the filament when the limiting curve is open are discussed throughout.

Key words. Stokes equations, resistance problems, mobility problems, layer potentials

AMS subject classifications. 76D07, 31B10, 35Q30, 74F10

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1. Introduction.

A small body or particle suspended in a viscous fluid will be transported from one location to another depending on the motion of the surrounding fluid, the action of external loads, and the size and shape of the body. An understanding of such transport, and of ways to control it, is fundamental to many areas of science and technology. Applications include the study of colloids, from the traditional theory [14, 22] to the emerging areas of actively-steered colloids [30] and magnetic swimmers [29], experimental methods for probing the structure of macromolecules such as proteins and DNA [3], and the modeling of various devices for the separation and analysis of particles in microfluidic systems [23], including the optimal design of microrobots in connection with targeted drug delivery [20], and more recently the design of devices for the capturing and counting of molecular markers of cancer and other diseases [17]. Indeed, the ability to drive a suspended body in intricate and programmable ways based on its shape has countless potential applications in biological and chemical analysis [33].

We consider the case in which the transport dynamics of a body are modeled by the Stokes equations for the viscous fluid in the exterior three-dimensional domain around the body, together with load balance relations for the quasi-static motion of the body within the fluid. The Stokes equations for the fluid are completed by a no-slip boundary condition on the surface of the body, and a prescribed flow far from the body. In this model the fluid motion is assumed to be nearly steady and slow, with small velocity gradients, so that inertial effects within the fluid can be ignored. Similarly, the body motion is assumed to be nearly steady and slow, so that the inertia of the body can also be ignored. Moreover, the body is assumed to be rigid, so that its associated equations of motion reduce to balance equations for the net force and torque. We note that a model for a single body in an infinite fluid is not only

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relevant for understanding the behavior of a rare or distinguished particle in a fluid sample, but also provides the basis for understanding the behavior of a distribution of particles at low concentrations [3].

We consider two basic types of transport problems for a body in a fluid with a prescribed far-field flow. In the first problem, the body velocities are given, and the external loads required to drive the body through the fluid are sought. In the second problem, the external loads are given, and the resulting velocities imparted to the body by the loads and fluid are sought. The general relation between the loads, velocities, and far-field flow in these problems is provided by the system of equations outlined above. Indeed, the relation is implicitly defined by the solution operator for the Stokes boundary-value problem in an exterior, three-dimensional domain. Closed-form solutions are generally unavailable, and numerical methods are required to approximate the transport properties of a body of a given shape [1, 2, 9, 26].

Here we focus our attention on bodies whose shape is that of a filament or tube, which refers to a cylindrical body of a given radius, whose axis is a general space curve. We study the Stokes equations in the exterior three-dimensional domain around such a body and consider the singular limit as the radius vanishes and the body collapses to a curve. We show that, under mild assumptions, the relation between transport velocities, external loads, and the far-field flow remains well defined and remarkably becomes explicit in this limit; theorems establishing the form of this relation in different cases are stated and proved. Hence the transport dynamics of an infinitesimally thin, rigid filament can be determined without direct consideration of the exterior Stokes boundary-value problem. We illustrate the results with two examples. In the first, we consider a body whose axial curve is straight (so the body is a cylinder) in a homogeneous far-field flow, and explicitly characterize the translational and rotational velocities imparted to the body by the flow in the infinitesimally thin or zero-radius limit. In the second, we consider a body whose axial curve is helical, and compare the transport properties in the zero-radius limit with those for positive radius obtained by a direct numerical treatment of the three-dimensional boundary-value problem.

The results derived here are of intrinsic mathematical interest and provide a contribution to the theory of slow viscous flow. While our results pertain only to the case of a vanishing radius, they may be useful in the study and design of bodies of small radius. For instance, the explicit relations derived herein may be useful as an initial approximation in optimal design and related problems for thin bodies. Also, here we only consider the case of a rigid body; the case of a flexible body is more difficult and outside the scope of the present work. The proof of our results is based on a tedious analysis of weakly singular integrals in a fully three-dimensional boundary integral formulation of the Stokes equations. Other approaches to our results could be contemplated. For instance, rather than begin from a fully three-dimensional formulation, one may instead begin from a reduced theory for Stokes flow around thin bodies, such as the resistive-force or slender-body theories [18, 19, 21, 24]. However, since these theories are only approximate, we opt to work with the exact three-dimensional theory to avoid any issues with approximation errors and their behavior in a singular limit. Indeed, the case when the axial curve of the body is an open arc, as opposed to a closed loop, gives rise to special considerations at the ends or extremities of the body. Various remarks on these issues will be made throughout. Although not pursued here, the relation between slender-body type theories and three-dimensional boundary integral formulations of Stokes flows is a topic of significant interest; see [24, 27, 28] and references therein.

2. Statement of main results. Here we outline the Stokes equations and introduce the flow quantities and notation that are needed to state the main results. For further background, see [14, 22, 25].

2.1. Stokes equations, domain. We consider the slow motion of a body in an incompressible viscous fluid in three-dimensional space. We denote the body domain by $D^- \subset \mathbb{R}^3$, the fluid domain by $D^+ \subset \mathbb{R}^3$, and the boundary between them by $\Gamma \subset \mathbb{R}^3$. The Stokes equations that describe the velocity field $u^+ : D^+ \to \mathbb{R}^3$ and pressure field $p^+ : D^+ \to \mathbb{R}$ of the fluid flow around the body are, in nondimensional form,

$$\begin{aligned} u^+_{i,jj}(x) &= p^+_{,i}(x), \\ u^+_{i,i}(x) &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \Delta u^+(x) &= \nabla p^+(x), \\ \nabla \cdot u^+(x) &= 0, \end{aligned} \qquad x \in D^+. \tag{2.1}$$

Equation $(2.1)_1$ is the local balance law of linear momentum for the fluid in the absence of an external force field, and $(2.1)_2$ is the local incompressibility constraint. We assume that $D^- \cup \Gamma \cup D^+$ fills all of three-dimensional space, that D^- and D^+ are both open and connected, and that D^- is bounded. Moreover, we assume that Γ is closed and bounded, and additionally is a Lyapunov surface [12], so that the techniques of classic potential theory for the Stokes equations may be applied. Briefly stated, a surface is Lyapunov if it has no self-intersections, is differentiable, and has a Hölder continuous outward unit normal field.

Unless mentioned otherwise, all vector quantities are referred to a single frame and indices take values from one to three. Moreover, we use the usual conventions that a pair of repeated indices implies summation, and that indices appearing after a comma denote partial derivatives. We assume here and throughout that all quantities have been nondimensionalized using a characteristic length scale $\ell > 0$, velocity scale $\vartheta > 0$, and force scale $\mu\vartheta\ell > 0$, where μ is the absolute viscosity of the fluid. The dimensional quantities corresponding to $\{x, u^+, p^+\}$ are $\{\ell x, \vartheta u^+, \mu\vartheta\ell^{-1}p^+\}$.

2.2. Boundary, decay conditions. For given fields $v : \Gamma \to \mathbb{R}^3$, $u^\infty : \mathbb{R}^3 \to \mathbb{R}^3$ and $p^\infty : \mathbb{R}^3 \to \mathbb{R}$, we consider the boundary conditions

$$\begin{aligned} &u^+_i(x) = v_i(x), && x \in \Gamma, \\ &u^+_i(x), p^+(x) \to u^\infty_i(x), p^\infty(x), && |x| \to \infty. \end{aligned} \tag{2.2}$$

Equation $(2.2)_1$ is a no-slip condition which states that the fluid and body velocities coincide at each point of the boundary, and $(2.2)_2$ is an asymptotic condition which states that the fluid velocity and pressure tend to prescribed values at infinity. We assume that the fields (u^∞, p^∞) are twice continuously differentiable and satisfy the Stokes equations (2.1) in all of space. Moreover, we use the notation in $(2.2)_2$ to denote the following decay conditions, under which existence and uniqueness results for the exterior Stokes system can be established [6, 15, 25]:

$$\begin{gathered} u^+_i(x) - u^\infty_i(x) = O(|x|^{-1}), \quad u^+_{i,j}(x) - u^\infty_{i,j}(x) = O(|x|^{-2}), \\ p^+(x) - p^\infty(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty. \end{gathered} \tag{2.3}$$

We assume that the body domain D^- is rigid and hence can only undergo rigid-body motion. In this case, the vector field v in $(2.2)_1$ takes the general form

$$v_i(x) = V_i + \varepsilon_{ijk}\Omega_j(x_k - c_k) \qquad \text{or} \qquad v(x) = V + \Omega \times (x - c), \tag{2.4}$$

where ε_{ijk} is the standard permutation symbol. Here $V \in \mathbb{R}^3$ is the linear velocity of a given reference point $c \in \mathbb{R}^3$, and $\Omega \in \mathbb{R}^3$ is the angular velocity of the body.

2.3. Stress, traction, loads. The stress field associated with a pair (u^+, p^+) is a function $\sigma^+ : D^+ \to \mathbb{R}^{3\times 3}$ defined by

$$\sigma^+_{ij}(x) = -p^+(x)\delta_{ij} + u^+_{i,j}(x) + u^+_{j,i}(x), \tag{2.5}$$

where δ_{ij} is the standard Kronecker delta symbol. For each $x \in D^+$ the stress tensor σ^+ is symmetric in the sense that $\sigma^+_{ij} = \sigma^+_{ji}$. To highlight the dependence of σ^+ on the pair (u^+, p^+) we use the notation $\sigma^+ = \sigma[u^+, p^+]$.

The traction field $h^+ : \Gamma \to \mathbb{R}^3$ exerted by the exterior fluid on the body surface (force per unit area) is defined by

$$h^+_i(x) = \sigma^+_{ij}(x)\nu_j(x), \tag{2.6}$$

where $\nu : \Gamma \to \mathbb{R}^3$ denotes the outward unit normal field. The resultant force $F \in \mathbb{R}^3$ and torque $T \in \mathbb{R}^3$, about the reference point c, associated with the traction field are

$$F_i = \int_\Gamma h^+_i(x)\, dA_x, \qquad T_i = \int_\Gamma \varepsilon_{ijk}(x_j - c_j)h^+_k(x)\, dA_x, \tag{2.7}$$

where dA_x denotes an infinitesimal area element at $x \in \Gamma$. To highlight the dependence of F and T on the traction field $h^+ = \sigma^+\nu$ and point c, we use the notation $F = F[h^+]$ and $T = T[h^+, c]$.

2.4. Load balance relations. When the body domain D^- is rigid and subject to external resultant loads $(F^{\text{ext}}, T^{\text{ext}})$, the slow quasi-static motion of the body through the fluid is described by the relations

$$F + F^{\text{ext}} = 0, \quad T + T^{\text{ext}} = 0. \tag{2.8}$$

Here (F, T) are the resultant force and torque of the fluid on the body as defined in (2.7). These relations express the balance laws of linear and angular momentum for the body, when inertial effects are ignored. These relations are assumed to hold for any choice of the reference point c, which serves as the reference for all loads and velocities as indicated in (2.7) and (2.4).

2.5. Basic problems. We consider two basic problems for the slow motion of a rigid body through a viscous fluid defined by (2.1)–(2.8).

The resistance problem. Given $(V, \Omega, u^\infty, p^\infty)$, find $(F^{\text{ext}}, T^{\text{ext}}, u^+, p^+)$. That is, given prescribed body velocities (V, Ω) and a prescribed far-field flow (u^∞, p^∞), find the external loads $(F^{\text{ext}}, T^{\text{ext}})$ required to propel the body and the resulting flow (u^+, p^+) in the surrounding vicinity.

The mobility problem. Given $(F^{\text{ext}}, T^{\text{ext}}, u^\infty, p^\infty)$, find (V, Ω, u^+, p^+). That is, given prescribed external loads $(F^{\text{ext}}, T^{\text{ext}})$ and a prescribed far-field flow (u^∞, p^∞), find the resulting body velocities (V, Ω) and the resulting flow (u^+, p^+) in the surrounding vicinity.

Existence and uniqueness results for both of the above problems can be established using potential theoretic techniques under the assumptions that Γ is closed, bounded, and Lyapunov; such results will be outlined later in Lemma 4.1. The representation of solutions in terms of potentials shows that the difference fields $u^+ - u^\infty$ and $p^+ - p^\infty$ will be smooth in D^+, but may possess only a finite number of bounded derivatives in $D^+ \cup \Gamma$ depending on the precise smoothness of the boundary Γ.

The *pure resistance* and *pure mobility* problems corresponding to the case of a vanishing far-field flow, $(u^\infty, p^\infty) = (0, 0)$, are of special interest. In this case, there is an invertible linear map between (V, Ω) and $(F^{\text{ext}}, T^{\text{ext}})$ that characterizes the drag properties of the body as it is driven through a resting fluid. In contrast, the *free mobility* problem corresponding to the case of vanishing external loads, $(F^{\text{ext}}, T^{\text{ext}}) = (0, 0)$, is also of special interest. In this case, there is a linear map from (u^∞, p^∞) to (V, Ω) that characterizes how a body is transported by a moving fluid.

The linear maps described above are functions of the body surface Γ and the reference point c, and are implicitly defined by the system of equations in (2.1)–(2.8). Below we show that exact, explicit expressions for these maps can be obtained for tubular surfaces in the limit of vanishing tube radius.

2.6. Main results. We consider a body whose bounding surface Γ_r is a uniform, cylindrical tube of radius $r > 0$. We suppose that Γ_r is centered on an axial curve $\gamma(s) \in \mathbb{R}^3$, which is parameterized by an arclength coordinate $s \in [-L, L]$, so that the total arclength is $2L > 0$. The curve γ may be open or closed; if open, we suppose that Γ_r is capped by hemispheres of radius r centered at the endpoints. The assumption of a uniform radius along the axis of the tube, along with hemispherical caps at the ends in the open case, is made for simplicity. Other types of tubular geometries could be considered, for example, the size and shape of the cross-sections could be nonuniform, and the caps at the ends could be nonhemispherical, and the surface could collapse onto its axial curve in different uniform and nonuniform ways, but such generalizations are not pursued here.

We consider the decomposition $\Gamma_r = \Gamma_{r,-} \cup \Gamma_{r,\text{crv}} \cup \Gamma_{r,+}$. Here $\Gamma_{r,\text{crv}}$ is the cylindrical portion of the surface corresponding to $\gamma(s)$ with $s \in (-L, L)$, and $\Gamma_{r,\pm}$ denotes the remaining portions of the surface associated with the points $\gamma(\pm L)$. Specifically, $\Gamma_{r,\pm}$ are hemispheres of radius r centered at $\gamma(\pm L)$ in the case when γ is open; otherwise, $\Gamma_{r,\pm}$ are (coincident) circles of radius r centered at $\gamma(\pm L)$ when γ is closed. We assume that γ is non-self-intersecting, except at the endpoints in the closed case, and that γ' is Lipschitz continuous. Among other things, these conditions imply that Γ_r is a well-defined Lyapunov surface for all $r \in (0, a_\gamma)$. Here $a_\gamma > 0$ denotes the injectivity radius (or global radius of curvature) of the curve γ [10]; it is defined as the supremum value of r for which the tubular surface Γ_r is non-self-intersecting.

We consider the system (2.1)–(2.8) for the surface Γ_r, with fixed data of either the resistance or mobility type as described above. For each $r \in (0, a_\gamma)$, the system is uniquely solvable, and there is a unique flow (u^+, p^+) in the exterior domain D_r^+ defined by Γ_r. This flow has a stress field σ^+ in D_r^+, and a traction field $h^+ = \sigma^+\nu$ on Γ_r. To state our results, it will be convenient to consider a rescaled traction field, which is defined on the cylindrical portion of the surface by

$$\Phi(r, x) = |r \ln(r)|\, h^+(x), \quad r \in (0, a_\gamma), \quad x \in \Gamma_{r,\text{crv}}, \tag{2.9}$$

and, in the open case, on the hemispherical caps by

$$\Phi(r, x) = r h^+(x), \quad r \in (0, a_\gamma), \quad x \in \Gamma_{r,-} \cup \Gamma_{r,+}. \tag{2.10}$$

The rescaled traction field Φ contains information on a family of solutions parameterized by $r \in (0, a_\gamma)$. We note that the factors of $r|\ln(r)|$ and r in the above expressions are distinguished scalings that reflect the growth rate of the traction field on the different portions of the surface in the singular limit as $r \to 0^+$. These scalings arise naturally in the analysis of the relevant boundary integral operators, and they can also be motivated based on local solutions of the Stokes equations around cylindrical and spherical geometries. In our developments, we will assume that the family of flows (u^+, p^+) parameterized by r is sufficiently regular in the following sense: the stress field σ^+ is continuous in $D_r^+ \cup \Gamma_r$ for each r, and the rescaled traction field Φ is bounded on the domain $\cup_r \Gamma_r$ and Hölder continuous on the domain $\cup_r \Gamma_{r,\mathrm{crv}}$. In the case when γ is open, different assumptions about the scaling of the traction on the end-caps in (2.10) could be considered; see the discussion after Lemma 4.4.

Given the axial curve γ and reference point c, we consider a matrix $G \in \mathbb{R}^{6\times 6}$ of the form

$$G = \begin{pmatrix} G_1 & G_3 \\ G_2 & G_4 \end{pmatrix}, \tag{2.11}$$

where $G_1, \ldots, G_4 \in \mathbb{R}^{3\times 3}$ are matrices defined by

$$\begin{gathered} G_1 = \int_{-L}^{L} g(s)\, ds, \quad G_2 = \int_{-L}^{L} [(\gamma(s) - c)\times] g(s)\, ds, \\ G_3 = \int_{-L}^{L} g(s)[(\gamma(s) - c)\times]^T\, ds, \quad G_4 = \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) [(\gamma(s) - c)\times]^T\, ds. \end{gathered} \tag{2.12}$$

Here $g(s) \in \mathbb{R}^{3\times 3}$ is defined by $g(s) = \mathrm{Id} - \frac{1}{2}(\gamma' \otimes \gamma')(s)$, where $\mathrm{Id} \in \mathbb{R}^{3\times 3}$ is the identity, $\gamma'(s) \in \mathbb{R}^3$ is the unit tangent to $\gamma(s)$, and $\otimes$ denotes the vector outer product; in components, we have $(a \otimes b)_{ij} = a_i b_j$ for any $a, b \in \mathbb{R}^3$. Moreover, for any vector $\eta \in \mathbb{R}^3$ we use the notation $[\eta\times] \in \mathbb{R}^{3\times 3}$ to denote the unique skew-symmetric matrix with the property that $[\eta\times]v = \eta \times v$ for all $v \in \mathbb{R}^3$; in components, we have $[\eta\times]_{ij} = \varepsilon_{ikj}\eta_k$, or more explicitly

$$[\eta\times] = \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}. \tag{2.13}$$

The matrix function g represents a simple, anisotropic scaling in directions that are normal and tangent to the curve γ; this type of scaling arises naturally from a local property of the fundamental solution of the Stokes equations, and has appeared in some of the earliest studies [4, 11, 13].

Our first result pertains to the free mobility problem for a body whose boundary is a tubular surface as described. The result shows that, under suitable conditions, the linear and angular velocities imparted to the body by a given far-field flow have a well-defined limit as the tube radius vanishes.

THEOREM 2.1. *Let an axial curve γ, reference point c, and far-field flow (u^∞, p^∞) be given. For any radius $r \in (0, a_\gamma)$, let (V_r, Ω_r) denote the body velocities in the free mobility problem for the tubular surface Γ_r. If γ' is Lipschitz continuous, σ^+ is continuous in $D_r^+ \cup \Gamma_r$, and Φ is bounded in $\cup_r \Gamma_r$ and Hölder continuous in $\cup_r \Gamma_{r,\mathrm{crv}}$, then*

$$\lim_{r\to 0^+} G \begin{pmatrix} V_r \\ \Omega_r \end{pmatrix} = \begin{pmatrix} \int_{-L}^{L} g(s) u^\infty(\gamma(s))\, ds \\ \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) u^\infty(\gamma(s))\, ds \end{pmatrix}. \tag{2.14}$$

Thus, under mild conditions, the product $G(V_r, \Omega_r)$ has a limit as the tube radius r vanishes, and the limiting value is completely characterized by the axial curve γ, reference point c, and far-field velocity u^∞. When the matrix G is invertible, the expression in (2.14) completely characterizes all components of $\lim_{r\to 0^+} V_r$ and

$\lim_{r\to 0^+} \varOmega_r$. On the other hand, when the matrix G is not invertible, the expression in (2.14) characterizes the components of $\lim_{r\to 0^+} V_r$ and $\lim_{r\to 0^+} \varOmega_r$ only up to the nullspace of G, and there is an associated compatibility condition on the data. As discussed below, G fails to be invertible only when γ is a line segment, and the system in (2.14) can be reduced to a simple form in this degenerate case.

Notice that G has a structure similar to that of a rotational inertia matrix for a material curve γ, with the symmetric, positive-definite matrix g playing the role of a mass density. Following this observation, it is straightforward to show that G is symmetric and positive-definite for arbitrary γ and c, provided that γ is not a line segment. In the degenerate case of a line segment, the matrix G is only semi-positive-definite, with a one-dimensional nullspace, and the compatibility condition associated with (2.14) is trivially satisfied.

The degenerate situation of a line segment can be illustrated by considering an orthonormal frame in which one of the basis vectors is parallel to the segment, and the reference point c is taken as the midpoint. In this case, the matrix g is constant and diagonal, G_2 and G_3 will vanish, G_1 will be diagonal with positive entries, and G_4 will be diagonal with two positive and one zero entry. The expression in (2.14) will determine all three components of $\lim_{r\to 0^+} V_r$, along with the two components of $\lim_{r\to 0^+} \varOmega_r$ perpendicular to the line; only the one component of $\lim_{r\to 0^+} \varOmega_r$ parallel to the line is undetermined. A separate, more detailed analysis would be required to characterize any limiting value of this component; however, since it plays no role in the kinematics of a line segment, we do not pursue that here.

The result in (2.14) characterizes the slow motion of an infinitesimally thin, rigid filament as it is freely transported by advection in a given flow u^∞. Specifically, for any given shape of the filament as described by the curve γ, and any given body reference point c, the vectors $V_0 := \lim_{r\to 0^+} V_r$ and $\varOmega_0 := \lim_{r\to 0^+} \varOmega_r$ would be the linear and angular velocities of the infinitesimally thin body about the point c. In view of (2.14) and (2.11), these velocities satisfy the relations

$$\begin{aligned} G_1 V_0 + G_3 \varOmega_0 &= \int_{-L}^{L} g(s) u^\infty(\gamma(s))\, ds, \\ G_2 V_0 + G_4 \varOmega_0 &= \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) u^\infty(\gamma(s))\, ds. \end{aligned} \tag{2.15}$$

Notice that these equations could be used to generate the translational and rotational trajectory of the body as it is transported by the flow, and that they can be assembled and solved for the body velocities without any explicit consideration of the exterior Stokes boundary-value problem outlined in (2.1)–(2.8).

The integrals on the right-hand side of (2.15) can be interpreted as weighted zeroth- and first-order moments of the velocity field u^∞ with respect to the curve γ and point c. It is interesting to note that V_0 and $\varOmega_0$ will, in general, depend on both of these moments, and it may not be possible to decouple the equations. Indeed, in view of (2.12) and the fact that g is not a scalar, there may be no choice of reference point c for which the off-diagonal blocks G_2 and G_3 vanish. One simple case in which the equations can be decoupled is the degenerate case of a line segment as outlined above; it is unlikely that the equations decouple in any other case.

Our second result pertains to the pure resistance problem for a body whose boundary is a tubular surface as described. The result shows that, under suitable conditions, the external force and torque required to propel the body with given linear and angular velocities vanish at well-defined rates as the tube radius vanishes.

THEOREM 2.2. *Let an axial curve γ, reference point c, and body velocities $(V, \varOmega)$ be given. For any radius $r \in (0, a_\gamma)$, let $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ denote the external loads in the pure resistance problem for the tubular surface $\varGamma_r$. If γ' is Lipschitz continuous, σ^+ is continuous in $D_r^+ \cup \varGamma_r$, and $\varPhi$ is bounded in $\cup_r \varGamma_r$ and Hölder continuous in $\cup_r \varGamma_{r,\mathrm{crv}}$, then*

$$\lim_{r\to 0^+} |\ln(r)| \begin{pmatrix} F_r^{\mathrm{ext}} \\ T_r^{\mathrm{ext}} \end{pmatrix} = 4\pi G \begin{pmatrix} V \\ \varOmega \end{pmatrix}. \tag{2.16}$$

Thus, under mild conditions, the product $|\ln(r)|(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ has a limit as the tube radius r vanishes, and the limiting value is completely characterized by the axial curve γ, reference point c, and body velocities $(V, \varOmega)$. Since $|\ln(r)| \to \infty$, it follows that $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}}) \to (0, 0)$ as $r \to 0^+$. The limiting path along which the force and torque vanish can be read directly from (2.16), namely $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}}) \to \frac{4\pi}{|\ln(r)|} G(V, \varOmega)$ as $r \to 0^+$.

In general, for the pure resistance and pure mobility problems, the external loads and body velocities are linearly related through a symmetric, positive-definite Stokes resistance matrix $R \in \mathbb{R}^{6\times 6}$ and mobility matrix $M = R^{-1} \in \mathbb{R}^{6\times 6}$ defined such that $(F^{\mathrm{ext}}, T^{\mathrm{ext}}) = R(V, \varOmega)$ and $(V, \varOmega) = M(F^{\mathrm{ext}}, T^{\mathrm{ext}})$ [14, 22]. For the case of a tubular surface as considered here, these matrices depend on the tube radius so that $R = R_r$ and $M = M_r$, and the result in (2.16) implies $R_r \to \frac{4\pi}{|\ln(r)|} G$ and, provided G is invertible, $M_r \to \frac{|\ln(r)|}{4\pi} G^{-1}$, as $r \to 0^+$.

The above results may be useful for comparing hydrodynamic properties of thin filaments. For example, the ratio of entries of R_r (resistance coefficients) or entries of M_r (mobility coefficients) could be compared between two different filament shapes, or a given filament shape to itself. In view of the above results, these ratios tend to well-defined values determined by the matrix G in the infinitesimally thin limit. We remark that this is a singular limit for the exterior Stokes boundary-value problem and would be difficult to access numerically.

A more general result that contains Theorems 2.1 and 2.2 as special cases can also be stated. It says that, if the limiting value of either velocities or loads are prescribed, then the limiting value of the other satisfies a simple relation that is explicit in the axial curve and reference point of the body.

THEOREM 2.3. *Let an axial curve γ, reference point c, and far-field flow (u^∞, p^∞) be given. For any radius $r \in (0, a_\gamma)$, let $(V_r, \varOmega_r)$ and $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ denote the body velocities and external loads in a resistance or mobility problem for the tubular surface $\varGamma_r$. Assume that γ' is Lipschitz continuous, σ^+ is continuous in $D_r^+ \cup \varGamma_r$, and $\varPhi$ is bounded in $\cup_r \varGamma_r$ and Hölder continuous in $\cup_r \varGamma_{r,\mathrm{crv}}$. If either of the limits $\lim_{r\to 0^+}(V_r, \varOmega_r)$ or $\lim_{r\to 0^+} |\ln(r)|(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ exists, then so does the other, and*

$$\begin{aligned} \lim_{r\to 0^+} \frac{|\ln(r)|}{4\pi} \begin{pmatrix} F_r^{\mathrm{ext}} \\ T_r^{\mathrm{ext}} \end{pmatrix} &= \lim_{r\to 0^+} G \begin{pmatrix} V_r \\ \varOmega_r \end{pmatrix} \\ &\quad - \begin{pmatrix} \int_{-L}^{L} g(s) u^\infty(\gamma(s))\, ds \\ \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) u^\infty(\gamma(s))\, ds \end{pmatrix}. \end{aligned} \tag{2.17}$$

The above result can be viewed as a fundamental limiting relation between the main quantities in a resistance or mobility problem: the far-field flow, body velocities, and external loads. Whereas the body velocities $(V_r, \varOmega_r)$ may achieve any finite value in the limit, the external loads $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ must necessarily vanish in order for $|\ln(r)|(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$ to achieve a finite value, and we note that this value would be zero for any loads which vanish faster than $1/|\ln(r)|$. This observation implies a certain stability or robustness for the general mobility problem: the result in Theorem 2.1 holds not only in the free case with prescribed loads $(F_r^{\rm ext}, T_r^{\rm ext}) \equiv (0,0)$ for all $r \in (0, a_\gamma)$, but also in more general cases with $(F_r^{\rm ext}, T_r^{\rm ext}) = o(1/|\ln(r)|)$ as $r \to 0^+$, for instance when the loads are proportional to the volume enclosed by Γ_r, or proportional to the surface area of Γ_r. For completeness, we note that the result in Theorem 2.2 for the resistance problem follows from the above relation when the velocities are fixed $(V_r, \Omega_r) \equiv (V, \Omega)$ for all $r \in (0, a_\gamma)$, and the far-field flow is assumed to be trivial, so that $u^\infty \equiv 0$. Alternatively, when $(V_r, \Omega_r) \equiv (0,0)$ and $u^\infty \not\equiv 0$, we note that (2.17) provides information on the external loads required to hold a body immobilized in a given far-field flow.

The proof of our results is based on a tedious analysis of weakly singular integrals in a fully three-dimensional boundary integral formulation of the Stokes equations. The main technical assumption is that of boundedness and Hölder continuity for the rescaled traction field Φ defined in (2.9) and (2.10), which is a growth and regularity assumption for the exterior Stokes problem in the singular limit as a three-dimensional body collapses to a curve. In particular, while the pointwise traction field may grow unbounded, we assume that a rescaled version of it is well behaved in the limit. Also, here we only consider the case of a rigid body; the case of a flexible body is more difficult and outside the scope of the present work.

We remark that the case of a three-dimensional body collapsing to a curve is rather special among a natural family of singular problems. For example, we could also consider a body collapsing in a well-defined way to a point or to a sheet. In the case of a point, the hydrodynamic properties of the limiting body would have a local and explicit dependence on the far-field flow: the limiting body would simply translate as a fluid particle. In the case of a sheet, the hydrodynamic properties of the limiting body would have a nonlocal and implicit dependence on the far-field flow and limiting geometry: the hydrodynamic properties would be characterized by a boundary-value problem similar to the case of a voluminous body. In contrast, the case of a body collapsing to a curve is special in the sense that the hydrodynamic properties of the limiting body has a nonlocal but completely explicit dependence on the far-field flow and limiting geometry.

Other approaches to our results could also be considered. For example, rather than begin from a fully three-dimensional formulation, one may instead begin from a reduced theory for Stokes flow around thin bodies, such as the resistive-force or slender-body theories [18, 19, 21, 24]. These theories provide an approximate relation between a pointwise force and a pointwise velocity distribution for a thin body of small radius. Resistive-force theory provides a simple local relation between the force and velocity distributions, whereas slender-body theory provides a nonlocal integral relation. In these approximate relations, the pointwise force and velocity distributions are assumed to depend only on the coordinate along the central axis of the body, and no-slip boundary conditions may or may not be satisfied at all points of, or even on, the body surface. While the approximate relations are expected to hold for thin bodies, there is, in general, no explicit knowledge of the errors, or of how these errors may be distributed along the central axis up to the ends. Indeed, the analysis of errors is delicate since the limit of vanishing radius is a singular limit for the exterior Stokes problem, and the resistive-force and slender-body approximations involve pointwise quantities that may only converge in a nonuniform or generalized (L_1) sense under this limit; see, for example, Lemmas 4.2–4.4, and the remark at the end of section 4.5. For these reasons we do not consider any reduced or approximate formulation. Instead, we establish all of our results by working directly with an exact, fully three-dimensional formulation.

3. Examples. Here we briefly illustrate the results in Theorems 2.1–2.3. We consider an infinitesimally thin, rigid filament with axial curve γ and reference point c in a far-field flow of the homogeneous form

$$u^\infty(x) = b + Ax, \quad p^\infty(x) \equiv 0. \tag{3.1}$$

Here $b \in \mathbb{R}^3$ is an arbitrary constant vector, and $A \in \mathbb{R}^{3\times3}$ is an arbitrary constant matrix satisfying $\mathrm{tr}(A) = 0$, as required by the divergence-free condition in (2.1). Using the same reference point c as the body, the above velocity field can be written in the equivalent form

$$u^\infty(x) = v^\infty + w^\infty \times (x - c) + S^\infty(x - c), \tag{3.2}$$

where $v^\infty = b + Ac$, $[w^\infty \times] = \frac{1}{2}(A - A^T)$, and $S^\infty = \frac{1}{2}(A + A^T)$. Notice that v^∞ corresponds to the local fluid velocity vector and hence streamline direction at the point c, whereas w^∞ corresponds to the constant rate of rotation (half of vorticity) vector throughout the homogeneous flow, and S^∞ corresponds to the constant rate of deformation (stretching and shearing) tensor.

Substitution of (3.2) into (2.17), and using the notation $V_0 := \lim_{r\to0^+} V_r$ and $\varOmega_0 := \lim_{r\to0^+} \varOmega_r$, we obtain

$$\begin{aligned}\lim_{r\to0^+} \frac{|\ln(r)|}{4\pi} \begin{pmatrix} F_r^{\text{ext}} \\ T_r^{\text{ext}} \end{pmatrix} &= G \begin{pmatrix} V_0 - v^\infty \\ \varOmega_0 - w^\infty \end{pmatrix} \\ &\quad - \begin{pmatrix} \int_{-L}^{L} g(s) S^\infty(\gamma(s) - c)\, ds \\ \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) S^\infty(\gamma(s) - c)\, ds \end{pmatrix}.\end{aligned} \tag{3.3}$$

For concreteness, we consider the free mobility problem in which the body is freely transported by advection in the absence of external loads, so that $(F_r^{\text{ext}}, T_r^{\text{ext}}) \equiv (0, 0)$, and $(V_0, \varOmega_0)$ are the velocities imparted to the body by the flow. In view of (3.3), these velocities satisfy

$$G \begin{pmatrix} V_0 - v^\infty \\ \varOmega_0 - w^\infty \end{pmatrix} = \begin{pmatrix} \int_{-L}^{L} g(s) S^\infty(\gamma(s) - c)\, ds \\ \int_{-L}^{L} [(\gamma(s) - c)\times] g(s) S^\infty(\gamma(s) - c)\, ds \end{pmatrix}. \tag{3.4}$$

When there is no stretching and shearing, so that $S^\infty = 0$ and the fluid flow is itself a rigid motion, we observe that the body velocities $(V_0, \varOmega_0)$ will match the corresponding fluid components (v^∞, w^∞). In this case, the body motion exactly matches the fluid motion as it is carried by the flow; in particular, the body reference point will follow a streamline. On the other hand, when stretching and shearing are present, so that $S^\infty \neq 0$, then $(V_0, \varOmega_0)$ and (v^∞, w^∞) will, in general, be different. In this case, the body motion cannot match the fluid motion as it is carried by the flow; in particular, the body reference point will generally not follow a streamline, regardless of how the reference point is chosen. An important exception occurs in the degenerate case when γ is a line segment and the reference point c is taken as the midpoint as outlined earlier. In this case, since g is constant and G is diagonal, we deduce that $V_0 = v^\infty$, so that the midpoint of a line segment will follow a streamline even when $S^\infty \neq 0$. However, for the rate of rotation vectors we deduce, in general, that $\varOmega_0 \neq w^\infty$ as a result of fluid shearing in the plane orthogonal to the line segment.

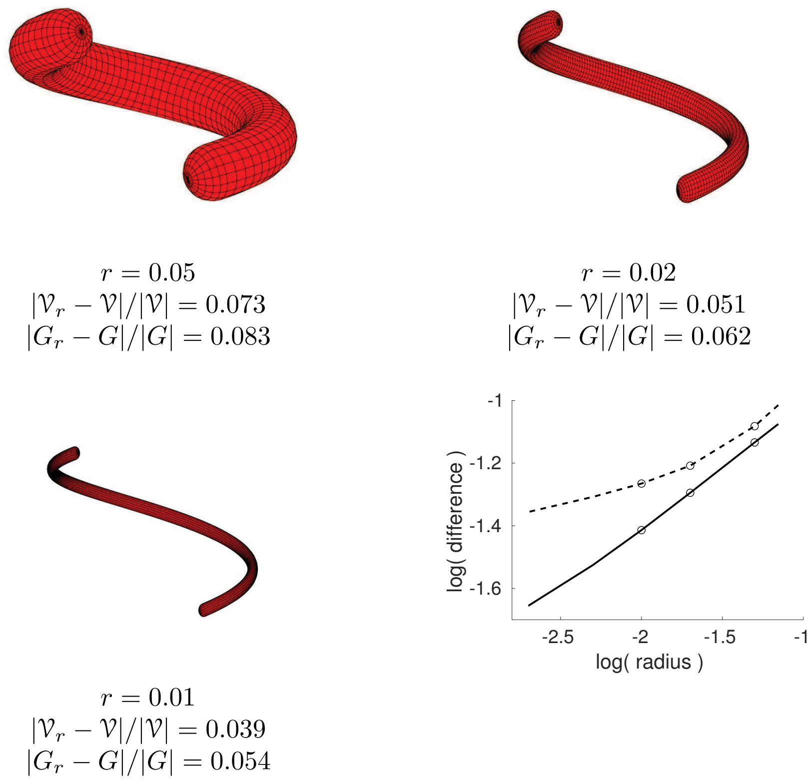

FIG. 3.1. *Numerical results for a body with a tubular surface centered on a helical curve. The free mobility velocity vector $\mathcal{V}_r$ and scaled resistance matrix G_r are compared against their zero-radius limits $\mathcal{V}$ and G for different values of the tube radius r. The bottom right panel shows plots of the relative differences $|\mathcal{V}_r - \mathcal{V}|/|\mathcal{V}|$ (solid) and $|G_r - G|/|G|$ (dashed) versus r, where $|\cdot|$ denotes the Euclidean norm; the three points marked in each plot correspond to $r = 0.05$, 0.02, and 0.01.*

The results in Theorems 2.1–2.3 provide exact results for the limiting case $r \to 0^+$. The theorems provide no information on the rates at which the limiting results are approached. While the analysis of such rates is outside the scope of the present work, we can nevertheless illustrate some aspects of the convergence by direct numerical treatment of the exterior Stokes boundary-value problem in (2.1)–(2.8). Figure 3.1 shows results for a body with tubular surface Γ_r centered on an axial curve γ corresponding to a helical arc, namely $\gamma(s) = (\alpha\cos(s/\eta), \alpha\sin(s/\eta), \beta s/\eta)$, $s \in [-L, L]$, where $\alpha = 0.2$ is the helical radius, $2\pi\beta = 0.3$ is the pitch, $2L = 1$ is the arclength and $\eta = \sqrt{\alpha^2 + \beta^2}$; all quantities are dimensionless as described in section 2.1. Using the reference point $c = (0, 0, 0)$, we considered the velocity vector $\mathcal{V}_r = (V_r, \Omega_r)$ for the free mobility problem in a far-field flow with $u^\infty(x) = (1 + x_2, 0, 0)$ and $p^\infty(x) \equiv 0$. Additionally, we considered the Stokes resistance matrix R_r [14, 22], along with the scaled matrix $G_r = \frac{|\ln(r)|}{4\pi} R_r$, associated with the pure resistance problem. In view of Theorem 2.1, we must have $\mathcal{V}_r \to \mathcal{V}$ as $r \to 0^+$, where $\mathcal{V} = (V_0, \Omega_0)$ is the limiting vector defined in (2.15). Also, in view of Theorem 2.2, we must have $G_r \to G$, where G is defined in (2.11). For decreasing values of the tube radius r, the vector $\mathcal{V}_r$ and matrix G_r were computed by a direct numerical treatment of (2.1)–(2.8) using a boundary element technique described elsewhere [7, 26]. The results show that for values of r in the range 0.07 to 0.002, the relative difference between $\mathcal{V}_r$ and $\mathcal{V}$ was in the range 8% to 2%, whereas the difference between G_r and G was in the range 10% to 4%. A plot of the relative differences versus radius suggests that the rates of convergence for $\mathcal{V}_r \to \mathcal{V}$ and $G_r \to G$ are rather slow. Whereas the rate for $G_r \to G$ appears to continually decrease in the direction of smaller radius, the rate for $\mathcal{V}_r \to \mathcal{V}$ appears to be nearly constant in this example. Due to the singular nature of the limit $r \to 0^+$, the direct treatment of the boundary-value problem becomes increasingly difficult, and analytical results such as those in Theorems 2.1–2.3 may provide the only practical means to explore the case of extremely small r. Moreover, the slow rates of convergence indicate that $\mathcal{V}, G$ provide only weak approximations to $\mathcal{V}_r, G_r$, thus a precise characterization of the differences $\mathcal{V}_r - \mathcal{V}$ and $G_r - G$ would be of significant practical utility. For recent results along these lines, see [27, 28].

4. Proof. Here we provide a proof of Theorems 2.1–2.3. We begin with some necessary facts about the Stokes single-layer potentials, and then present a series of lemmas which lead to the main theorems. We first state all the results, and then outline the proofs. When a result holds for an arbitrary body surface, we will use the notation Γ to denote the surface, and use D^- and D^+ to denote the associated interior and exterior domains. When a result is specialized to a tubular surface of radius r as outlined in section 2.6, we will use the notation Γ_r to denote the surface, and use D_r^- and D_r^+ to denote the associated interior and exterior domains. As before, we omit indices on vector and tensor quantities whenever there is no cause for confusion.

4.1. Preliminaries. Let $\psi : \Gamma \to \mathbb{R}^3$ be a continuous function. Then by the Stokes single-layer velocity and pressure potentials on Γ with density ψ we mean

$$
\begin{aligned}
U_i[\Gamma,\psi](x) &= \int_\Gamma E^{ij}(x,y)\psi_j(y)\,dA_y,\\
P[\Gamma,\psi](x) &= \int_\Gamma \Pi^j(x,y)\psi_j(y)\,dA_y.
\end{aligned}
\tag{4.1}
$$

Here (E^{ij}, Π^j) is a fundamental solution of the Stokes equations referred to as a stokeslet; it is a solution of the free-space equations with a singular (Dirac) force at the point y [7, 32]. Using the notation $z = x - y$, and $|\cdot|$ for the Euclidean norm, an explicit expression for this solution is

$$
E^{ij}(x,y) = \frac{\delta_{ij}}{|z|} + \frac{z_i z_j}{|z|^3}, \quad \Pi^j(x,y) = \frac{2z_j}{|z|^3}.
\tag{4.2}
$$

We remark that, due to the linearity of the free-space equations, the above solution is defined up to an arbitrary choice of normalization. The choice of normalization naturally affects various constants in the developments that follow, but is not crucial in any way; the choice adopted here is taken from [7].

For any continuous density ψ, the potentials $(U[\Gamma,\psi], P[\Gamma,\psi])$ are smooth at each $x \notin \Gamma$. Moreover, by virtue of their definitions as a linear combination of stokeslets, they satisfy the Stokes equations (2.1) at each $x \notin \Gamma$. While we consider the Stokes potentials with a density in the space of continuous functions, they could also be considered on various Sobolev spaces [16], but such generality will not be exploited here. The velocity potential $U[\Gamma,\psi]$ is finite for all $x \in D^- \cup \Gamma \cup D^+$. In the special case when $x \in \Gamma$, the corresponding integral is only weakly singular, and hence exists as an improper integral in the usual sense [12] provided that Γ is a Lyapunov surface. The restriction of $U[\psi, \Gamma]$ to Γ is denoted by $\overline{U}[\psi, \Gamma]$. This restriction is a continuous function on Γ; moreover, for any $x_0 \in \Gamma$, the following pointwise limit relations hold [25]:

$$\lim_{\substack{x \to x_0 \\ x \in D^+}} U[\Gamma, \psi](x) = \overline{U}[\Gamma, \psi](x_0), \tag{4.3}$$

$$\lim_{\substack{x \to x_0 \\ x \in D^-}} U[\Gamma, \psi](x) = \overline{U}[\Gamma, \psi](x_0). \tag{4.4}$$

Standard arguments [12] can be used to show that both of the above limits converge uniformly in $x_0 \in \Gamma$.

4.2. Lemmata. We begin by summarizing a solvability result for the resistance and mobility problems outlined in section 2.5 for the exterior Stokes system in (2.1)–(2.8). Various forms and special cases of the result have been described elsewhere [8, 16, 25, 31]. Here we consider a form that is convenient for our purposes.

LEMMA 4.1. *Let a closed, bounded Lyapunov surface Γ, a reference point c, and a k-times ($k \geq 2$) continuously differentiable far-field flow (u^∞, p^∞) be given. Then the resistance problem and mobility problem for Γ are each uniquely solvable, and the unique flow (u^+, p^+) is k-times continuously differentiable in D^+. Assuming $\sigma^+ = \sigma[u^+, p^+]$ is continuous up to Γ, the fields (u^+, p^+) can be represented by single-layer potentials with a continuous density ψ in the form*

$$u^+(x) = u^\infty(x) + U[\Gamma, \psi](x), \quad p^+(x) = p^\infty(x) + P[\Gamma, \psi](x), \quad x \in D^+. \tag{4.5}$$

Moreover, the above representation holds with

$$\psi(x) = -\frac{1}{8\pi} h^+(x), \quad x \in \Gamma, \tag{4.6}$$

and implies the boundary integral relation

$$\frac{1}{8\pi} \int_\Gamma E(x, y) h^+(y)\, dA_y = u^\infty(x) - V - \Omega \times (x - c), \quad x \in \Gamma. \tag{4.7}$$

We next examine the single-layer integral in (4.7). Before considering a general tubular surface Γ_r as described in section 2.6, we first consider an open, straight cylindrical surface $\Gamma_{r,\mathrm{str}}$ and examine the matrix-valued function

$$\mathcal{E}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\mathrm{str}}} E(x, y)\, dA_y, \quad x \in \Gamma_{r,\mathrm{str}}. \tag{4.8}$$

We will be interested in characterizing the limit of this function as $r \to 0^+$. As we will see, the results obtained for this function will be fundamental.

To begin, let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, and consider the axial curve $\gamma(s) = s e_3$, where $s \in [-L, L]$, and the surface defined by

$$\begin{aligned} \Gamma_{r,\mathrm{str}} = \{x \in \mathbb{R}^3 \mid\ & x = r\cos\theta\, e_1 + r\sin\theta\, e_2 + s\, e_3, \\ & \qquad 0 \leq \theta < 2\pi,\ -L < s < L\}. \end{aligned} \tag{4.9}$$

Given two points $x, y \in \Gamma_{r,\mathrm{str}}$ we use the notation $x = x_r(\theta_x, s_x)$ and $y = y_r(\theta_y, s_y)$ to denote their representations in the coordinates (θ, s). Notice that $x_r(\theta_x, s_x) \to \gamma(s_x)$

and $y_r(\theta_y, s_y) \to \gamma(s_y)$ uniformly on $[0, 2\pi) \times (-L, L)$ as $r \to 0^+$. At any point $y = y_r(\theta_y, s_y)$ we can parameterize an infinitesimal area element as $dA_y = r d\theta_y ds_y$. In view of (2.12) and the fact that $\gamma'(s_x) \equiv e_3$, we consider the constant, diagonal matrices

$$g_{\text{str}} = \text{diag}\left(1, 1, \frac{1}{2}\right), \quad g_{\text{str}}^{-1} = \text{diag}(1, 1, 2). \tag{4.10}$$

Lemma 4.2. *Let $\mathcal{E}$ denote the single-layer integral (4.8) for the straight cylindrical surface $\Gamma_{r,\text{str}}$. For each $x = x_r(\theta_x, s_x)$ on $\Gamma_{r,\text{str}}$ we have the pointwise result*

$$\lim_{r \to 0^+} \frac{1}{|r \ln(r)|} \mathcal{E}(x_r(\theta_x, s_x)) = \frac{1}{2} g_{\text{str}}^{-1}, \quad (\theta_x, s_x) \in [0, 2\pi) \times (-L, L). \tag{4.11}$$

The above limit converges in the L_1-norm. Finite jumps occur at the endpoints $s_x = \pm L$ in the sense that

$$\lim_{r \to 0^+} \frac{1}{|r \ln(r)|} \mathcal{E}(x_r(\theta_x, s_x)) = \frac{1}{4} g_{\text{str}}^{-1}, \quad (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \tag{4.12}$$

The above result shows that $\mathcal{E}(x_r(\theta_x, s_x))/|r \ln(r)|$ converges to a function in the L_1-norm as $r \to 0^+$, with the function being constant on the domain $(\theta_x, s_x) \in [0, 2\pi) \times (-L, L)$. In particular, we have $\mathcal{E}(x_r(\theta_x, s_x)) \to 0$ and $|r \ln(r)| \to 0$, with a well-defined ratio in the limit. The factor of $|r \ln(r)|$ arises naturally in the analysis due to the cylindrical geometry and the definition of the stokeslet function $E(x, y)$ in (4.2).

The result in Lemma 4.2 can be generalized to the case of an open, curved cylindrical surface $\Gamma_{r,\text{crv}}$. To state the result, let $\gamma(s) \in \mathbb{R}^3$ be a given axial curve, where $s \in [-L, L]$ is an arclength parameter. At each point $\gamma(s)$, let $\{d_1(s), d_2(s), d_3(s)\}$ be a given orthonormal frame or basis for $\mathbb{R}^3$, defined such that $d_3(s) = \gamma'(s)$, so that $d_1(s)$ and $d_2(s)$ are perpendicular to the curve at each point. This local frame is needed only to parameterize the cylindrical surface, and is allowed to twist around $\gamma'(s)$ in an arbitrary way; for simplicity, we suppose the frame is twist-free as detailed below. Within this setup we consider the surface defined by

$$\begin{aligned} \Gamma_{r,\text{crv}} = \{x \in \mathbb{R}^3 \mid \ & x = r\cos\theta\, d_1(s) + r\sin\theta\, d_2(s) + \gamma(s), \\ & 0 \le \theta < 2\pi,\ -L < s < L\}. \end{aligned} \tag{4.13}$$

Similar to before, given two points $x, y \in \Gamma_{r,\text{crv}}$, we use the notation $x = x_r(\theta_x, s_x)$ and $y = y_r(\theta_y, s_y)$ to denote their representations in the coordinates (θ, s). At any point $y = y_r(\theta_y, s_y)$ we can parameterize an infinitesimal area element as $dA_y = J_r(\theta_y, s_y)\, r d\theta_y ds_y$, where $J_r(\theta_y, s_y)$ is a Jacobian factor given by

$$J_r(\theta_y, s_y) = 1 - (u \cdot d_2)(s_y) r \cos\theta_y + (u \cdot d_1)(s_y) r \sin\theta_y. \tag{4.14}$$

Here $u(s) \in \mathbb{R}^3$ is the Darboux vector (angular velocity in s) for the local frame, which by definition satisfies the kinematical equations

$$d_i'(s) = u(s) \times d_i(s), \quad s \in (-L, L), \quad i = 1, 2, 3. \tag{4.15}$$

The constraint that $d_3(s) = \gamma'(s)$ implies $u(s) = \gamma'(s) \times \gamma''(s) + \beta(s)\gamma'(s)$, where $\beta(s) \in \mathbb{R}$ corresponds to an arbitrary twist rate around $\gamma'(s)$. For convenience, we suppose the given frame is twist-free in the sense that $\beta(s) \equiv 0$. Thus u is determined by the curve γ, and the local frame d_i is determined by the integration of (4.15) up to an arbitrary choice of frame at $s = 0$.

We assume that γ' is Lipschitz continuous, which implies that u is defined almost everywhere and uniformly bounded, and hence the frame vectors d_i are Lipschitz continuous. We note that $\Gamma_{r,\mathrm{crv}}$ is a well-defined Lyapunov surface for all $r \in (0, a_\gamma)$, where $a_\gamma > 0$ is the injectivity radius of γ. Similar to before, we note that $x_r(\theta_x, s_x) \to \gamma(s_x)$, $y_r(\theta_y, s_y) \to \gamma(s_y)$, and $J_r(\theta_y, s_y) \to 1$ uniformly on $[0, 2\pi)\times(-L, L)$ as $r \to 0^+$. In view of (2.12), we consider the matrices

$$g(s) = \mathrm{Id} - \frac{1}{2}(\gamma' \otimes \gamma')(s), \quad g^{-1}(s) = \mathrm{Id} + (\gamma' \otimes \gamma')(s), \tag{4.16}$$

and analogous to (4.8), we consider the integral

$$\mathcal{E}(x) = \frac{1}{8\pi}\int_{\Gamma_{r,\mathrm{crv}}} E(x, y)\, dA_y, \quad x \in \Gamma_{r,\mathrm{crv}}. \tag{4.17}$$

Lemma 4.3. *Let $\mathcal{E}$ denote the single-layer integral (4.17) for the curved cylindrical surface $\Gamma_{r,\mathrm{crv}}$ with axial curve γ, and assume that γ' is Lipschitz continuous. For each $x = x_r(\theta_x, s_x)$ on $\Gamma_{r,\mathrm{crv}}$ we have the pointwise result*

$$\lim_{r\to 0^+} \frac{1}{|r\ln(r)|}\mathcal{E}(x_r(\theta_x, s_x)) = \frac{1}{2}g^{-1}(s_x), \quad (\theta_x, s_x) \in [0, 2\pi) \times (-L, L). \tag{4.18}$$

The above limit converges in the L_1-norm. When γ is open, finite jumps occur at the endpoints $s_x = \pm L$ in the sense that

$$\lim_{r\to 0^+} \frac{1}{|r\ln(r)|}\mathcal{E}(x_r(\theta_x, s_x)) = \frac{1}{4}g^{-1}(s_x), \quad (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \tag{4.19}$$

When γ is closed, the limit in (4.18) holds at all points.

Thus, just as before, the ratio $\mathcal{E}(x_r(\theta_x, s_x))/|r\ln(r)|$ converges to a function in the L_1-norm as $r \to 0^+$, but now the function is generally not constant on the domain $(\theta_x, s_x) \in [0, 2\pi)\times(-L, L)$. Instead, the limiting function depends on the unit tangent field along the axial curve as encapsulated in the matrix $g^{-1}(s)$. The above results reduce to those in Lemma 4.2 in the case when the surface is a straight cylinder, with axial curve $\gamma(s) = se_3$ and unit tangent $\gamma'(s) \equiv e_3$.

We now return to (4.7) and state a final technical result for the single-layer integral

$$S(x) = \frac{1}{8\pi}\int_{\Gamma_r} E(x, y)h^+(y)\, dA_y, \quad x \in \Gamma_r. \tag{4.20}$$

We consider a general tubular surface Γ_r with axial curve γ as described in section 2.6. The curve γ may be open or closed; if open, we suppose that Γ_r is capped by hemispheres of radius r centered at the endpoints. In either case, we consider the decomposition $\Gamma_r = \Gamma_{r,-} \cup \Gamma_{r,\mathrm{crv}} \cup \Gamma_{r,+}$, where $\Gamma_{r,\mathrm{crv}}$ is as defined in (4.13), and $\Gamma_{r,\pm}$ denote the remaining portions of the surface as previously described. Following (2.9) and (2.10), we consider a rescaled traction field, which is defined on the cylindrical portion of the surface by

$$\Phi(r, x) = |r\ln(r)|\, h^+(x), \quad r \in (0, a_\gamma), \quad x \in \Gamma_{r,\mathrm{crv}}, \tag{4.21}$$

and, in the open case, on the hemispherical caps by

$$\Phi(r, x) = rh^+(x), \quad r \in (0, a_\gamma), \quad x \in \Gamma_{r,-} \cup \Gamma_{r,+}. \tag{4.22}$$

Moreover, for points $x = x_r(\theta_x, s_x)$ on $\mathit{\Gamma}_{r,\mathrm{crv}}$, we introduce the circumferential average $\overline{\mathit{\Phi}}(r, s_x)$ defined by

$$\overline{\mathit{\Phi}}(r, s_x) = \frac{1}{2\pi}\int_0^{2\pi} \mathit{\Phi}(r, x_r(\theta, s_x))\, d\theta. \tag{4.23}$$

Last, let $g(s)$ and $g^{-1}(s)$ denote the matrices given in (4.16).

Lemma 4.4. *Let S denote the single-layer integral (4.20) for the tubular surface $\mathit{\Gamma}_r$ with axial curve γ, and assume that γ' is Lipschitz continuous. If the traction function $\mathit{\Phi}$ is bounded in $\cup_r \mathit{\Gamma}_r$ and Hölder continuous in $\cup_r \mathit{\Gamma}_{r,\mathrm{crv}}$, then for each $x = x_r(\theta_x, s_x)$ on $\mathit{\Gamma}_{r,\mathrm{crv}}$ we have*

$$\lim_{r\to 0^+} S(x_r(\theta_x, s_x)) = \frac{1}{2} g^{-1}(s_x)\overline{\mathit{\Phi}}_0(s_x), \quad (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \tag{4.24}$$

where $\overline{\mathit{\Phi}}_0(s_x)$ is the bounded, continuous function defined by

$$\overline{\mathit{\Phi}}_0(s_x) = \lim_{r\to 0^+} \overline{\mathit{\Phi}}(r, s_x). \tag{4.25}$$

The first limit above converges in the L_1-norm, whereas the second converges uniformly. The pointwise limit in (4.24) remains bounded at the endpoints $s_x = \pm L$ and finite jumps may occur, depending on whether the axial curve γ is open or closed. The pointwise limit in (4.25) can be extended continuously to $s_x = \pm L$.

In view of Lemmas 4.2 and 4.3, the factor of $|r \ln(r)|$ for the traction field in (4.21) is a distinguished scaling that arises naturally on the cylindrical subset $\mathit{\Gamma}_{r,\mathrm{crv}}$. Since $h^+ = \mathit{\Phi}/|r\ln(r)|$ and $\mathit{\Phi}$ is assumed to be bounded, we note that the scale factor represents the maximum growth rate for the traction field on $\mathit{\Gamma}_{r,\mathrm{crv}}$ as $r \to 0^+$. When the axial curve γ is open, the factor of r in (4.22) is a scaling that also arises naturally, but other scalings could be assumed, leading to slightly different versions of Lemma 4.4. For instance, the factor of r could be replaced with r^α, and for $\alpha < 2$ the same results in (4.24) and (4.25) would hold, with convergence in the same norms, but at the expense of unbounded pointwise limits in (4.24) at the endpoints $s_x = \pm L$. A more general scaling such as r^α on $\mathit{\Gamma}_{r,\pm}$ might arise under different assumptions about the ends in the open case, but would not affect the results stated in Theorems 2.1–2.3 provided that $\alpha < 2$; we remark that end-effects may dominate, and a different analysis would be required if $\alpha \geq 2$. In contrast, when the axial curve γ is closed, there are no ends to consider, and the scaling in (4.21) is natural at all points.

4.3. Proof of Lemmas 4.1–4.3. Here we briefly outline the essential ideas in the proofs of Lemmas 4.1–4.3. Full details of the proofs are given in the Supplementary Material (SMM129205.pdf [local/web 47KB]); they are omitted here for reasons of space.

The solvability result in Lemma 4.1 is obtained by introducing the shifted fields $w^+ = u^+ - u^\infty$ and $q^+ = p^+ - p^\infty$, and considering the decomposition $w^+ = \widetilde{w}^+ - \widehat{w}^+$ and $q^+ = \widetilde{q}^+ - \widehat{q}^+$, where $(\widetilde{w}^+, \widetilde{q}^+)$ and $(\widehat{w}^+, \widehat{q}^+)$ satisfy the exterior boundary-value problems

$$\begin{array}{llll} & \Delta\widetilde{w}^+ = \nabla\widetilde{q}^+, & \Delta\widehat{w}^+ = \nabla\widehat{q}^+, & x \in D^+, \\ & \nabla\cdot\widetilde{w}^+ = 0, & \nabla\cdot\widehat{w}^+ = 0, & x \in D^+, \\ (4.26) & \widetilde{w}^+ = V + \mathit{\Omega}\times(x - c), & \widehat{w}^+ = u^\infty, & x \in \mathit{\Gamma}, \\ & \widetilde{w}^+, \widetilde{q}^+ \to 0, 0, & \widehat{w}^+, \widehat{q}^+ \to 0, 0, & |x| \to \infty. \end{array}$$

Well-known potential theory results [5, 7, 8, 16, 25, 31, 32] for the systems in (4.26) imply that the second system is uniquely solvable and generates well-defined resultant loads on $\varGamma$ by regularity of the far field velocity; moreover, due to the rigid body form of the data, the first system is also uniquely solvable when either velocities or resultant loads on $\varGamma$ are prescribed. The solvability of a general resistance or mobility problem then follows by superposition. The representation result in Lemma 4.1 is obtained by considering an auxiliary interior problem for fields (w^-, q^-), namely

$$
\begin{aligned}
&\Delta w^- = \nabla q^-, && x \in D^-,\\
&\nabla \cdot w^- = 0, && x \in D^-,\\
&w^- = V + \varOmega \times (x - c) - u^\infty, && x \in \varGamma.
\end{aligned} \tag{4.27}
$$

Well-known results for the exterior problem, based on the fundamental solution of the Stokes equations and the divergence theorem, imply that the solution of the systems in (4.26) and hence the overall exterior fields (w^+, q^+) possess a natural representation involving both the single- and double-layer Stokes potentials. Similarly, since the condition $\int_\varGamma w^- \cdot \nu \, dA_x = 0$ is satisfied, the interior fields (w^-, q^-) also possess such a representation. By combining the representations for (w^+, q^+) and (w^-, q^-), and using the fact that $w^+ = w^-$ on $\varGamma$, the double-layer terms can be eliminated to obtain a purely single-layer representation as claimed.

The result in Lemma 4.2 is obtained by combining (4.2) with (4.8) and considering the function

$$
\mathcal{E}(x) = \mathcal{I}(x) + \mathcal{A}(x), \tag{4.28}
$$

where

$$
\mathcal{I}_{ij}(x) = \frac{1}{8\pi} \int_\varGamma \frac{\delta_{ij}}{|x-y|} \, dA_y, \quad \mathcal{A}_{ij}(x) = \frac{1}{8\pi} \int_\varGamma \frac{(x-y)_i (x-y)_j}{|x-y|^3} \, dA_y. \tag{4.29}
$$

At the moment, we consider the case when $\varGamma = \varGamma_{r,\mathrm{str}}$, which is a straight cylindrical surface as described in section 4.2. By properties of weakly singular integrals, the components $\mathcal{I}_{ij}(x)$ and $\mathcal{A}_{ij}(x)$ are well defined and continuous functions of $x \in \varGamma_{r,\mathrm{str}}$ for each $r > 0$, and we seek to characterize their limits as $r \to 0^+$. The result follows from an examination of the single independent component of $\mathcal{I}_{ij}$, and the six independent components of $\mathcal{A}_{ij}$. Working in cylindrical coordinates, and scaling the axial coordinate by r, we find that each of these component surface integrals can be transformed into a line integral of an elliptic-type function, and its pointwise limit as $r \to 0^+$ can be explicitly investigated. Specifically, we find that each of the components $\frac{1}{r}\mathcal{I}_{ij}$ and $\frac{1}{r}\mathcal{A}_{ij}$ either remains bounded, or diverges at a rate proportional to $|\ln(r)|$ as $r \to 0^+$. This result establishes the distinguished nature of the scaling $|r\ln(r)|$, and we obtain a pointwise limit result for each of the components $\frac{1}{|r\ln(r)|}\mathcal{I}_{ij}$ and $\frac{1}{|r\ln(r)|}\mathcal{A}_{ij}$. A dominating function is then identified for each of these components to establish L_1-convergence via the dominated convergence theorem.

To obtain the result in Lemma 4.3, we again consider the functions in (4.28) and (4.29), but with $\varGamma = \varGamma_{r,\mathrm{crv}}$, which is a curved cylindrical surface with axial curve γ. In this case, integrals over $\varGamma_{r,\mathrm{crv}}$ can be transformed into integrals over $\varGamma_{r,\mathrm{str}}$ using a natural mapping between these two surfaces. A key observation is that, for values of r below the injectivity radius of γ, this mapping satisfies a two-sided or bi-Lipschitz type bound, so that the chord length between a pair of points on $\varGamma_{r,\mathrm{crv}}$ is uniformly bounded above and below by the chord length between the corresponding pair on $\varGamma_{r,\mathrm{str}}$. Hence the resulting integrals have the same weakly singular integrands as considered before, up to bounded multiplicative factors with well-defined limits as $r \to 0^+$. The results for $\mathit{\Gamma}_{r,\mathrm{crv}}$ then follow from those for $\mathit{\Gamma}_{r,\mathrm{str}}$. The result is first established under the assumption that the curve γ is open; however, aside from the assumption of a positive injectivity radius, the result relies only on local properties of the curve, and hence also applies to the case when γ is closed.

4.4. Proof of Lemma 4.4. We consider the decomposition $\mathit{\Gamma}_r = \mathit{\Gamma}_{r,-} \cup \mathit{\Gamma}_{r,\mathrm{crv}} \cup \mathit{\Gamma}_{r,+}$, where $\mathit{\Gamma}_{r,\mathrm{crv}}$ is as defined in (4.13), and $\mathit{\Gamma}_{r,\pm}$ denote the remaining portions of the surface associated with the points $\gamma(\pm L)$ as previously described. We assume that the axial curve γ of $\mathit{\Gamma}_{r,\mathrm{crv}}$ is open; the case when γ is closed will be discussed later. We also consider the rescaled traction field $\mathit{\Phi}(r,x)$, defined in (4.21) and (4.22) for each $r \in (0, a_\gamma)$ and $x \in \mathit{\Gamma}_r$. Moreover, we consider the circumferential average $\overline{\mathit{\Phi}}(r, s_x)$, defined in (4.23) for each point $x = x_{r,\mathrm{crv}}(\theta_x, s_x)$ on $\mathit{\Gamma}_{r,\mathrm{crv}}$. The assumption that $\mathit{\Phi}(r,x)$ is bounded in $\cup_r \mathit{\Gamma}_r$ and Hölder continuous in $\cup_r \mathit{\Gamma}_{r,\mathrm{crv}}$ implies that, for every $r \in (0, a_\gamma)$ and $\widetilde{r} \in (0, a_\gamma)$, and $y \in \mathit{\Gamma}_r$, $x \in \mathit{\Gamma}_{r,\mathrm{crv}}$, and $\widetilde{x} \in \mathit{\Gamma}_{\widetilde{r},\mathrm{crv}}$, we have

$$|\mathit{\Phi}(r,y)| \le C, \qquad |\mathit{\Phi}(r,x) - \mathit{\Phi}(\widetilde{r},\widetilde{x})| \le C|x - \widetilde{x}|^\lambda, \tag{4.30}$$

where $C > 0$ and $0 < \lambda < 1$ are fixed constants.

From (4.30) we deduce some useful results for the circumferential average $\overline{\mathit{\Phi}}(r, s_x)$. Specifically, for every $r \in (0, a_\gamma)$, and $s_x \in (-L, L)$ and $s_{\widehat{x}} \in (-L, L)$, we find by straightforward arguments that

$$|\mathit{\Phi}(r,x) - \overline{\mathit{\Phi}}(r,s_x)| \le Cr^\lambda, \qquad |\overline{\mathit{\Phi}}(r,s_x) - \overline{\mathit{\Phi}}(r,s_{\widehat{x}})| \le C|s_x - s_{\widehat{x}}|^\lambda. \tag{4.31}$$

Also, since it is bounded and uniformly continuous on $\cup_r \mathit{\Gamma}_{r,\mathrm{crv}}$, the function $\mathit{\Phi}(r,x)$ can be extended with the same properties to the closure of this domain, which includes all points $x = x_{r,\mathrm{crv}}(\theta_x, s_x)$ under the limit $r \to 0^+$. From this we can deduce that the function defined by

$$\overline{\mathit{\Phi}}_0(s_x) = \lim_{r \to 0^+} \overline{\mathit{\Phi}}(r, s_x) \tag{4.32}$$

is bounded and continuous for $s_x \in (-L, L)$, and the above limit converges uniformly. Moreover, $\overline{\mathit{\Phi}}_0(s_x)$ can be extended continuously to $s_x = \pm L$.

We next consider the single-layer integral $S(x)$ in (4.20). Using the decomposition $\mathit{\Gamma}_r = \mathit{\Gamma}_{r,-} \cup \mathit{\Gamma}_{r,\mathrm{crv}} \cup \mathit{\Gamma}_{r,+}$, and the definition of the rescaled traction field $\mathit{\Phi}(r,x)$ in (4.21) and (4.22), we have

$$\begin{aligned} S(x) = {} & \frac{1}{8\pi |r \ln(r)|} \int_{\mathit{\Gamma}_{r,\mathrm{crv}}} E(x,y)\mathit{\Phi}(r,y)\, dA_y \\ & + \frac{1}{8\pi r^\alpha} \int_{\mathit{\Gamma}_{r,-} \cup \mathit{\Gamma}_{r,+}} E(x,y)\mathit{\Phi}(r,y)\, dA_y. \end{aligned} \tag{4.33}$$

For generality, we use a scaling of r^α in place of r in (4.22), and examine the effect of the exponent $\alpha > 0$; the case of interest is $\alpha = 1$. The above is a well-defined and continuous function of $x \in \mathit{\Gamma}_r$ for $r > 0$. For each point on the cylindrical subset $\mathit{\Gamma}_{r,\mathrm{crv}}$, we seek to characterize the limit as $r \to 0^+$.

We consider the first term in (4.33). To begin, let $x = x_{r,\mathrm{crv}}(\theta_x, s_x) \in \mathit{\Gamma}_{r,\mathrm{crv}}$ be given, and consider the decomposition

$$\frac{1}{8\pi |r \ln(r)|} \int_{\mathit{\Gamma}_{r,\mathrm{crv}}} E(x,y)\mathit{\Phi}(r,y)\, dA_y = S^{(0)}(x) + S^{(1)}(x) + S^{(2)}(x), \tag{4.34}$$

where

$$S^{(0)}(x) = \frac{1}{8\pi|r\ln(r)|}\int_{\Gamma_{r,\mathrm{crv}}} E(x,y)[\Phi(r,y) - \overline{\Phi}(r,s_y)]\, dA_y, \tag{4.35}$$

$$S^{(1)}(x) = \frac{1}{8\pi|r\ln(r)|}\int_{\Gamma_{r,\mathrm{crv}}} E(x,y)[\overline{\Phi}(r,s_y) - \overline{\Phi}(r,s_x)]\, dA_y, \tag{4.36}$$

$$S^{(2)}(x) = \frac{1}{8\pi|r\ln(r)|}\int_{\Gamma_{r,\mathrm{crv}}} E(x,y)\overline{\Phi}(r,s_x)\, dA_y. \tag{4.37}$$

For the term $S^{(0)}(x)$, we use the first inequality in (4.31), together with the bound $|E(x,y)| \le C/|x-y|$, which follows from (4.2), and the function $\mathcal{I}_{11}(x)$ from (4.29) with $\Gamma = \Gamma_{r,\mathrm{crv}}$, to get

$$|S^{(0)}(x)| \le \frac{Cr^{\lambda}\mathcal{I}_{11}(x)}{|r\ln(r)|}. \tag{4.38}$$

The function $\frac{1}{|r\ln(r)|}\mathcal{I}_{11}(x)$ was considered in the proof of Lemma 4.3, where it was shown to have a bounded pointwise limit for each $(\theta_x, s_x) \in [0,2\pi)\times(-L,L)$, including $s_x = \pm L$, and converge in the L_1-norm. For the function $S^{(0)}(x)$ we then obtain, in view of the factor Cr^{λ}, the pointwise limit $\lim_{r\to 0^+} S^{(0)}(x_{r,\mathrm{crv}}(\theta_x, s_x)) = 0$ for $(\theta_x, s_x) \in [0,2\pi)\times(-L,L)$, including $s_x = \pm L$, and we note that this convergence holds in the L_1-norm. For the term $S^{(1)}(x)$, we use the second inequality in (4.31), together with the bounds $|E(x,y)| \le C/|x-y|$ and $|s_x - s_y| \le |x-y|$, to get

$$|S^{(1)}(x)| \le \frac{C}{|r\ln(r)|}\int_{\Gamma_{r,\mathrm{crv}}} |s_x - s_y|^{\lambda-1}\, dA_y \le \frac{C}{|\ln(r)|}, \tag{4.39}$$

where the second inequality above follows from a direct integration, using the fact that the Jacobian in (4.14) is bounded. From this we obtain the pointwise limit $\lim_{r\to 0^+} S^{(1)}(x_{r,\mathrm{crv}}(\theta_x, s_x)) = 0$ for $(\theta_x, s_x) \in [0,2\pi)\times(-L,L)$, including $s_x = \pm L$, which converges uniformly. For the term $S^{(2)}(x)$, we can move $\overline{\Phi}(r,s_x)$ outside of the integral and use (4.17) to write

$$S^{(2)}(x) = \frac{1}{|r\ln(r)|}\mathcal{E}(x)\overline{\Phi}(r,s_x). \tag{4.40}$$

Using the result of Lemma 4.3 and the fact that $\overline{\Phi}(r,s_x) \to \overline{\Phi}_0(s_x)$ uniformly, we obtain the pointwise limit $\lim_{r\to 0^+} S^{(2)}(x_{r,\mathrm{crv}}(\theta_x, s_x)) = \frac{1}{2}g^{-1}(s_x)\overline{\Phi}_0(s_x)$ for $(\theta_x, s_x) \in [0,2\pi)\times(-L,L)$, which converges in the L_1-norm. And from Lemma 4.3 we note that the pointwise limit would be $\frac{1}{4}g^{-1}(s_x)\overline{\Phi}_0(s_x)$ at $s_x = \pm L$.

We next consider the second term in (4.33), and decompose the integral into two parts corresponding to $\Gamma_{r,+}$ and $\Gamma_{r,-}$. To obtain a result for $\Gamma_{r,+}$, we temporarily shift the origin of coordinates to the point $\gamma(L)$, and we let $\Sigma_{1,+}$ be the unit hemisphere corresponding to $\Gamma_{r,+}$, and Σ_1 the unit sphere; all centered at the origin. In this way, we obtain the parameterization $y = r\xi$, where $y \in \Gamma_{r,+}$ and $\xi \in \Sigma_{1,+}$, and the area element relation $dA_y = r^2 dA_\xi$. For any point $x \in \Gamma_{r,\mathrm{crv}}$ we then get, using the inequalities $|\Phi(r,x)| \le C$ and $|E(x,y)| \le C/|x-y|$, and the inclusion $\Sigma_{1,+} \subset \Sigma_1$,

$$\begin{aligned} &\frac{1}{8\pi r^{\alpha}}\left|\int_{\Gamma_{r,+}} E(x,y)\Phi(r,y)\, dA_y\right| \\ &\qquad \le \frac{C}{r^{\alpha}}\int_{\Gamma_{r,+}} \frac{1}{|x-y|}\, dA_y \le Cr^{1-\alpha}\int_{\Sigma_1} \frac{1}{|(x/r)-\xi|}\, dA_\xi. \end{aligned} \tag{4.41}$$

By properties of weakly singular integrals, the function $f(z) := \int_{\Sigma_1} \frac{1}{|z-\xi|}\, dA_\xi$ is continuous and uniformly bounded for all $z \in \mathbb{R}^3$; indeed, it is harmonic in the exterior of Σ_1, and is constant on and within the interior of Σ_1. It also satisfies the bound $f(z) \leq C/|z|$ for all $|z| \neq 0$, so that $f(z) \to 0$ as $|z| \to \infty$. We next consider any point $x = x_{r,\mathrm{crv}}(\theta_x, s_x)$ on $\Gamma_{r,\mathrm{crv}}$, and for convenience we temporarily shift the parameter interval from $s_x \in (-L, L)$ to $s_x \in (-2L, 0)$. Using the fact that γ' is Lipschitz continuous, together with the fact that γ has a positive injectivity radius, we get the pointwise bound $f(x_{r,\mathrm{crv}}(\theta_x, s_x)/r) \leq Cr/\sqrt{r^2 + s_x^2}$. Moreover, by direct integration over (θ_x, s_x), we get the L_1-norm bound $||f|| \leq C|r\ln(r)|$. We can now examine the right-hand side of (4.41), which takes the form $Cr^{1-\alpha} f(x_{r,\mathrm{crv}}(\theta_x, s_x)/r)$. Provided that $2 - \alpha > 0$, the right-hand side of (4.41) vanishes for each $s_x \in (-2L, 0)$ as $r \to 0^+$, and moreover, the L_1-norm vanishes. At the endpoint $s_x = 0$, we have the bound $Cr^{1-\alpha} f(x_{r,\mathrm{crv}}(\theta_x, 0)/r) \leq Cr^{2-\alpha}/\sqrt{r^2} = Cr^{1-\alpha}$, which implies the pointwise limit at $s_x = 0$ vanishes if $1 - \alpha > 0$, must be bounded if $\alpha = 1$, and may be unbounded if $1 - \alpha < 0$. Note that identical results would be obtained if $\Gamma_{r,+}$ is replaced by $\Gamma_{r,-}$ in (4.41), and the origin is temporarily shifted to $\gamma(-L)$. From this we deduce that the second term in (4.33) has a vanishing limit in the L_1-norm as $r \to 0^+$ provided that $\alpha < 2$. Furthermore, the pointwise limits at the endpoints will vanish if $\alpha < 1$, must be bounded if $\alpha = 1$, and may be unbounded if $\alpha > 1$.

By combining the above results for the terms in (4.33) we obtain the following limit for each point $x = x_{r,\mathrm{crv}}(\theta_x, s_x) \in \Gamma_{r,\mathrm{crv}}$, which converges in the L_1-norm,

$$\lim_{r\to 0^+} S(x_{r,\mathrm{crv}}(\theta_x, s_x)) = \frac{1}{2} g^{-1}(s_x)\overline{\Phi}_0(s_x), \quad (\theta_x, s_x) \in [0, 2\pi) \times (-L, L). \tag{4.42}$$

In establishing the above result, we assumed that the curve γ was open with a Lipschitz unit tangent field, and also non-self-intersecting, so that its injectivity radius was positive. However, similar to the proof of Lemma 4.3, we note that the above result relies only on local properties of γ and $\overline{\Phi}_0$, and hence also applies to the case when the curve is closed, provided it has a Lipschitz unit tangent at the closure point, and is non-self-intersecting except for the closure point. In this case, the closure point is not special in any way, and the value of the limit is $\frac{1}{2} g^{-1}(s_x)\overline{\Phi}_0(s_x)$ for all s_x.

4.5. Proof of Theorems 2.1–2.3. Consider either of the resistance or mobility problems for the tubular surface Γ_r, with body velocities (V_r, Ω_r) and external loads $(F_r^{\mathrm{ext}}, T_r^{\mathrm{ext}})$. Under the assumptions of Theorems 2.1–2.3, the resistance and mobility problems for Γ_r are uniquely solvable as established in Lemma 4.1, and the boundary-integral relation in (4.7) holds. And this relation can be expressed in terms of the integral $S(x)$ in (4.20), and the rescaled traction field $\Phi(r, x)$ in (4.21) and (4.22).

For concreteness, we suppose the axial curve γ is open; when γ is closed, the situation is more straightforward since $\Gamma_{r,\pm}$ would then have zero measure, and hence no contribution in the developments below due to the boundedness of the rescaled traction. From (2.7) and (2.8) we have

$$F_r^{\mathrm{ext}} = -\int_{\Gamma_r} h^+(x)\, dA_x, \qquad T_r^{\mathrm{ext}} = -\int_{\Gamma_r} (x - c) \times h^+(x)\, dA_x. \tag{4.43}$$

Making use of the decomposition $\Gamma_r = \Gamma_{r,-} \cup \Gamma_{r,\mathrm{crv}} \cup \Gamma_{r,+}$, employing spherical coordinates (θ, ϕ) in each of the hemispherical caps $\Gamma_{r,-}$ and $\Gamma_{r,+}$ with area element $dA_x = r^2 \sin\phi_x d\theta_x d\phi_x$, along with curvilinear cylindrical coordinates (θ, s) in $\Gamma_{r,\mathrm{crv}}$ with area element $dA_x = J_r(\theta_x, s_x)\, r d\theta_x ds_x$, where $J_r(\theta_x, s_x)$ is as defined in (4.14), and introducing the traction function $\Phi(r,x)$ following (4.21) and (4.22), we get

$$
\begin{aligned}
-|\ln(r)|F_r^{\text{ext}} = &\int_{\Gamma_{r,-}\cup\Gamma_{r,+}} \Phi(r,x)\ r^{2-\alpha}|\ln(r)|\sin\phi_x d\theta_x d\phi_x \\
&+ \int_{\Gamma_{r,\text{crv}}} \Phi(r,x)J_r(\theta_x,s_x)\ d\theta_x ds_x
\end{aligned} \tag{4.44}
$$

and

$$
\begin{aligned}
-|\ln(r)|T_r^{\text{ext}} = &\int_{\Gamma_{r,-}\cup\Gamma_{r,+}} (x-c)\times\Phi(r,x)\ r^{2-\alpha}|\ln(r)|\sin\phi_x d\theta_x d\phi_x \\
&+ \int_{\Gamma_{r,\text{crv}}} (x-c)\times\Phi(r,x)J_r(\theta_x,s_x)\ d\theta_x ds_x.
\end{aligned} \tag{4.45}
$$

As before, for generality, we have used a scaling of r^α in place of r in (4.22), and in view of the discussion following Lemma 4.4 we suppose $\alpha < 2$. Using the fact that $\Phi(r,x)$ is bounded uniformly for $r \in (0, a_\gamma)$ and $x \in \Gamma_r$, and that $J_r(\theta_x, s_x) \to 1$ and $x = x_{r,\text{crv}}(\theta_x, s_x) \to \gamma(s_x)$ and $\overline{\Phi}(r, s_x) \to \overline{\Phi}_0(s_x)$ uniformly in (θ_x, s_x) as $r \to 0^+$, where $\overline{\Phi}(r, s_x)$ is the circumferential average of $\Phi(r,x)$ on $\Gamma_{r,\text{crv}}$ considered in Lemma 4.4, we deduce that

$$
\begin{aligned}
&\lim_{r\to 0^+} -|\ln(r)|F_r^{\text{ext}} = 2\pi\int_{-L}^{L} \overline{\Phi}_0(s_x)\ ds_x, \\
&\lim_{r\to 0^+} -|\ln(r)|T_r^{\text{ext}} = 2\pi\int_{-L}^{L} (\gamma(s_x)-c)\times\overline{\Phi}_0(s_x)\ ds_x.
\end{aligned} \tag{4.46}
$$

Combining (4.7) and (4.20), and restricting attention to points $x = x_{r,\text{crv}}(\theta_x, s_x)$ on $\Gamma_{r,\text{crv}}$ we have

$$
S(x_{r,\text{crv}}(\theta_x,s_x)) = u^\infty(x_{r,\text{crv}}(\theta_x,s_x)) - V_r - \Omega_r\times(x_{r,\text{crv}}(\theta_x,s_x)-c). \tag{4.47}
$$

Multiplying (4.47) by the matrix $g(s_x)$, integrating over (θ_x, s_x), and using the fact that $x = x_{r,\text{crv}}(\theta_x, s_x) \to \gamma(s_x)$ uniformly in (θ_x, s_x) as $r \to 0^+$, and also that $S(x_{r,\text{crv}}(\theta_x, s_x)) \to \frac{1}{2}g^{-1}(s_x)\overline{\Phi}_0(s_x)$ in the L_1-norm in (θ_x, s_x) as $r \to 0^+$ as established in Lemma 4.4, which holds for any $\alpha < 2$, we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{-L}^{L}\overline{\Phi}_0(s_x)\ ds_x = &\int_{-L}^{L} g(s_x)u^\infty(\gamma(s_x))\ ds_x \\
&- \int_{-L}^{L} g(s_x)V_0\ ds_x - \int_{-L}^{L} g(s_x)[(\gamma(s_x)-c)\times]^T\Omega_0\ ds_x.
\end{aligned} \tag{4.48}
$$

Here we use the notation V_0 and Ω_0 to indicate the limits of V_r and Ω_r as $r \to 0^+$. In a similar way, multiplying (4.47) by the matrix $[(\gamma(s_x) - c)\times]g(s_x)$, we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{-L}^{L}(\gamma(s_x)-c)\times\overline{\Phi}_0(s_x)\ ds_x = &\int_{-L}^{L}[(\gamma(s_x)-c)\times]g(s_x)u^\infty(\gamma(s_x))\ ds_x \\
&- \int_{-L}^{L}[(\gamma(s_x)-c)\times]g(s_x)V_0\ ds_x \\
&- \int_{-L}^{L}[(\gamma(s_x)-c)\times]g(s_x)[(\gamma(s_x)-c)\times]^T\Omega_0\ ds_x.
\end{aligned} \tag{4.49}
$$

Combining the above two equations with (4.46), and using the matrix notation introduced in (2.11), we obtain

$$\lim_{r\to 0^+} \frac{|\ln(r)|}{4\pi} \begin{pmatrix} F_r^{\text{ext}} \\ T_r^{\text{ext}} \end{pmatrix} = \lim_{r\to 0^+} G \begin{pmatrix} V_r \\ \Omega_r \end{pmatrix} - \begin{pmatrix} \int_{-L}^{L} g(s) u^\infty(\gamma(s))\, ds \\ \int_{-L}^{L} [(\gamma(s)-c)\times] g(s) u^\infty(\gamma(s))\, ds \end{pmatrix}. \tag{4.50}$$

The result in Theorem 2.3 follows from (4.50), and the results in Theorems 2.1 and 2.2 follow as special cases. For the free mobility problem considered in Theorem 2.1, the external loads are fixed so that $(F_r^{\text{ext}}, T_r^{\text{ext}}) \equiv (0,0)$ for all r, and we obtain the result in (2.14). For the pure resistance problem considered in Theorem 2.2, the far-field flow vanishes so that $u^\infty \equiv 0$, and the body velocities are fixed so that $(V_r, \Omega_r) \equiv (V, \Omega)$ for all r, and we obtain the result in (2.16).

As a final remark, we make an observation about the relation in (4.47), which holds for $x = x_{r,\text{crv}}(\theta_x, s_x)$ on $\Gamma_{r,\text{crv}}$ with $r \in (0, a_\gamma)$. For the left-hand side, we note that $S(x_{r,\text{crv}}(\theta_x, s_x)) \to \frac{1}{2} g^{-1}(s_x)\overline{\Phi}_0(s_x)$ as $r \to 0^+$, where $\overline{\Phi}_0(s)$ can be interpreted as a limiting, rescaled force distribution along the axial curve $\gamma(s)$. Hence (4.47) implies a pointwise relation between a local force-like quantity and a velocity akin to those considered in resistive-force and slender-body theories. We remark that this limiting pointwise relation has restricted applicability, since the left-hand side may converge only in the L_1-norm. Moreover, in view of the discussion following Lemma 4.4, the pointwise limit of the left-hand side may be unbounded at the endpoints in the open case when more general scalings r^α are considered in (4.22). For rigid bodies, we only consider moments of the relation in (4.47), which are well-defined under L_1-convergence.

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