MATH 328 (Rusin) — FINAL EXAM — due May 15 2020 at 11:59pm

This final is to be taken following the same rules we used for Exam 2. The details may be found at

`http://web.ma.utexas.edu/users/rusin/328K-20a/final-rules`

Make sure you write out your answers, scan them, and upload them to Canvas by Friday at 11:59pm Austin time.

Each question is worth 10 points. Please answer each question in full on its own page (but please do not turn in your illegible "scratch" work). Remember to prove all your statements – sometimes a string of equations is enough, but usually there need to be enough words to show how the equations are supposed to connect logically to support your claims.

1. What is the largest power of 12 that divides $18^{24} + 24^{18}$?

2. The *Tribonacci numbers* T_n are defined by $T_1 = 0, T_2 = 0, T_3 = 1$. and for $n > 3$ by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. Show that for all $n > 0$ we have $T_n \leq 2^{n-3}$.

3. (a) Prove that for every $n > 0$, the sum of the first n even natural numbers is $n^2 + n$.

(b) The number $a = 41$ has the following remarkable property. First of all, this a is prime. If we add 2 to a, we get another prime (43). If we add 4 to this, we get another prime (47). Continue in this way, each time adding the next even number. Every single one of the numbers attained in this way is prime until the composite 40th term $(1681 = 41^2)$

Prove that no matter which number $a > 0$ is used for the start of this sequence, we will always eventually encounter a composite number.

4. Show that if a and b are two coprime positive integers, then no odd prime p can divide both $a + b$ and $a^2 + b^2$.

5. Show that if a is any integer coprime to 561, then $a^{560} \equiv 1 \pmod{561}$.

6. Find all integer solutions (x, y) to *one* of the following equations

$$(a)\ 4x + 9y = 35 \qquad (b)\ 6x^2 + xy - 12y^2 = 35 \qquad (c)\ 3x^2 - y^2 = 35$$

For extra credit, find all integer solutions to either or both of the other equations.

7. Show that if $x \equiv 4 \pmod 9$ or $x \equiv 5 \pmod 9$ then it is impossible to write x as a sum of three perfect cubes (positive, negative, or zero).

8. Use Quadratic Reciprocity to determine whether or not 73 is a square modulo 59.

9. Is 16 a cube modulo $p_1 = 2000387$? I will tell you that p_1 is prime and 2 is a primitive root modulo p_1; that may help. Answer the same question with the prime modulus $p_2 = 6001747$. for which 2 is again a primitive root.

10. If we should be forced to hold our Fall 2020 classes online, how should we do it? What worked this semester — in our class or any other class you took — and what did not work?