Math 341 (Rusin) — EXAM 2 ANSWERS — Nov 8 2018.

Here is an answer key to your second exam. For each question I will give both an answer (or two) that you could have given, together with some commentary about the answers you did give, and about how you could have reacted more fruitfully to the question when you saw it.

A word about the answers I propose here: When I explain something in class, I say a lot and only write a little. In that case, the *explanation* is what comes out of my mouth; what I write is just a summary, or an outline, of the proof. When you are giving a proof on paper, no one is speaking: what is written has to be the entirety of the proof. All I can do when grading is to read exactly what's on the paper, starting at the top. If you don't write words, there are no words. So you can't just write things like:

$$\text{“ } a = b. \qquad c = d. \qquad e = f. \text{ ”;}$$

because I don't know if you mean "since $a = b$ and $c = d$, we conclude $e = f$" or "if $a = b$, then $c = d$ and $e = f$" (or whatever), and those sentences mean very different things! So when I present an answer to the problems below, I am proposing what a student *should literally write on the exam,* including all the words and symbols you see, arranged as shown. Obviously you will have your own mathematical writing style — more wordy or less wordy, etc. — but I'm giving you real, live samples of what good student responses should look like. When the instructions are "find", or "compute", or "calculate", then a bunch of equations are probably enough; but when the instructions are "show" or "explain" or "prove", it's got to be something I could read aloud. OK?

OK, let's go.

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1. Compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix}$$

ANSWER:

$$p_A(x) = \det(A - xI) = \det(\begin{pmatrix} 1-x & 2 & 0 & 0 \\ 3 & 4-x & 0 & 0 \\ 0 & 0 & 5-x & 6 \\ 0 & 0 & 7 & 8-x \end{pmatrix}).$$

Expand along the top row:

$$= +(x-1)\det(\begin{pmatrix} 4-x & 0 & 0 \\ 0 & 5-x & 6 \\ 0 & 7 & 8-x \end{pmatrix}) - 2\det(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5-x & 6 \\ 0 & 7 & 8-x \end{pmatrix}) - 0 + 0$$

Expand each along its top row to get

$$= +(x-1)(4-x)\det(\begin{pmatrix} 5-x & 6 \\ 7 & 8-x \end{pmatrix}) - 2\cdot 3\det(\begin{pmatrix} 5-x & 6 \\ 7 & 8-x \end{pmatrix})$$

$$= \big((x-1)\cdot(x-4) - 2\cdot 3\big) \times \big((x-5)\cdot(x-8) - 6\cdot 7\big)$$

$$= (x^2-5x-2)(x^2-13x-2) = x^4 - 18x^3 + 61x^2 + 36x + 4.$$

COMMENTS: Observe that the characteristic polynomial is the product of the characteristic polynomials of the smaller 2×2 matrices. This is true in general: if a matrix can be written in blocks as $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B and C are smaller square matrices, then $\det(A) = \det(B)\det(C)$, and so we conclude $p_A(x) = p_B(x)p_C(x)$.

You can compute the determinants of the 2×2 or even the 3×3 matrices shown by those "diagonal rules" you know, but as I emphasized in class, there is **no** such "rule" for larger matrices. A 4×4 determinant is a sum/difference of $4! = 24$ products of various entries of the matrix, not 8 of them.

My calculus students know $\log(xy) = \log(x) + \log(y)$ and keep hoping there is a way to rewrite $\log(x+y)$ too. There isn't. You guys, similarly, know that $\det(AB) = \det(A)\det(B)$ and apparently hope there is a way to rewrite $\det(A+B)$. There isn't. That's why we have to write the characteristic polynomial as $\det(xI - A)$ and not as something simpler, like, say, $\det(A) - x^n$ or something. That's just wrong; don't do it.

2. Suppose that B is a 3×3 matrix and I is the 3×3 identity matrix. Suppose the real number c is an eigenvalue of B. Show that $c-1$ is an eigenvalue of $B-I$.

I'll give a little extra credit if you can tell me something about the eigen*vectors* of $B-I$, too.

ANSWER 1: If c is an eigenvalue of B then c is a root of the characteristic polynomial $p_B(x)$, i.e. $p_B(c) = 0$. Thus $\det(cI-B) = 0$. But $cI - B = cI - I - B + I = (c-1)I - (B-I)$ so $\det(cI-B) = \det((c-1)I - (B-I)$, that is, $p_B(c) = p_{B-I}(c-1)$. So $p_B(c) = 0 \Rightarrow p_{B-I}(c-1) = 0$. Hence $c-1$ is a root of $p_{B-I}(x)$, the characteristic polynomial of $B-I$, which means $c-1$ is an eigenvalue of $B-I$.

ANSWER 2: If c is an eigenvalue of B then there must be corresponding eigenvector $v \neq 0$ with $Bv = cv$. Then $(B-I)v = Bv - Iv = cv - v = (c-1)v$, which is to say that this same vector v is also an eigenvector for $B-I$, and shows that $c-1$ is an eigenvalue of $B-I$.

COMMENTS: Obviously I like the second proof better because it also gives the extra credit answer! The eigenvectors of $B-I$ are exactly the same vectors as the eigenvectors of B. But the first answer is fine, too, as long as you do say how the eigenvalues are related to the characteristic polynomial. (I was looking for the word "root" or something about p_B being zero.) And again, don't be thinking that $\det(xI - (B-I)) = \det(xI - B + I)$ can be simply evaluated as $\det(xI-B) + \det(I)$ or some such thing!

If you didn't know what to do on this problem it might have paid off to toy around with an actual matrix B for which you know an eigenvalue, like maybe $B = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$, which has 6 as an eigenvalue. Try to figure out "why" $B - I$ has 5 as an eigenvalue. One example does not a proof make, but it might give you some insight that leads to one of the two answers above.

3. Suppose C is a square matrix. Show that C is singular (i.e. non-invertible) iff 0 is an eigenvalue of C.

ANSWER: If C is singular then $\det(C) = 0$, i.e. $0 = \det(C - 0I) = p_C(0)$. So 0 is a root of the characteristic polynomial, meaning it's an eigenvalue of C.

If instead C is nonsingular, then C has an inverse, C^{-1}. So now if 0 were an eigenvalue of C, there would be an eigenvector to go with it: $Cv = 0v$ which (on the right) is the zero vector. Hence $v = C^{-1}Cv = C^{-1}(0) = 0$, contrary to the fact that an eigenvector is supposed to be a *nonzero* vector. This contradiction shoes that 0 cannot be an eigenvalue of C when C is nonsingular.

COMMENTS: Notice that there are two paragraphs above: there should always be, when you're proving an "iff" statement; one paragraph proves "$\Rightarrow$" and the other proves the converse.

Instead of my second paragraph above, you could more or less read the first paragraph backwards; that would be an alternative second-half proof.

You could in this problem have used any of the characterizations of what the word "singular" means (p.172). What does it mean to *you?* For example, you might have proved the first half of the answer like this: if C is singular, then (see the table) there is a non-trivial solution to the linear system $CX = 0$. Well, then that X is a vector in $\mathbf{R}^n$ and $CX = 0X$, so X is a 0-eigenvector, meaning 0 is an eigenvalue, QED.

4. The matrix

$$F = \begin{pmatrix} 3 & 2 & 8 \\ 8 & 7 & 8 \\ 7 & 6 & 8 \end{pmatrix}$$

has -1 as an eigenvalue. (You don't have to prove this.) Find an eigenvector for this eigenvalue.

ANSWER: Need a v with $Fv = (-1)v$ i.e. $(F - (-1)I)v = 0$. So need to solve this linear system:

$$\begin{pmatrix} 4 & 2 & 8 & \vdots & 0 \\ 8 & 8 & 8 & \vdots & 0 \\ 7 & 6 & 9 & \vdots & 0 \end{pmatrix}$$

[*Evidence of row-reduction omitted here because I'm too lazy to type it but it should end* *with:*]

$$\begin{pmatrix} 1 & 0 & 3 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Solution: $\begin{pmatrix} -3z \\ 2z \\ z \end{pmatrix}$ for any z. So an eigenvector is $[-3, 2, 1]$.

COMMENTS: Any multiple of this is also a (-1)-eigenvector, of course.

You really should then compute Fv and make sure that it does equal $(-1)v$: it's so much faster to *check* your eigenvalues than it is to compute them in the first place, so why would you not check your own work?

5. Is this a vector space? Explain.

$$V = \{(x, y, z) \in \mathbf{R}^3 \mid z = 1\}$$

where we define

$$(x, y, 1) \oplus (x', y', 1) = (x + x', y + y', 1) \qquad \text{and} \qquad r \odot (x, y, 1) = (rx, ry, 1)$$

ANSWER: Yes, V is a vector space. We prove this by checking the definition of "vector space".

The definitions of the two operations clearly show that the $\oplus$-sum of any two elements of V is back in V, and likewise for $\odot$, that is, V is closed under the two operations.

Now we check the 8 axioms.

(1) $(x, y, 1) \oplus (x', y', 1) = (x + x', y + y', 1) = (x' + x, y' + y, 1) = (x', y', 1) \oplus (x, y, 1)$

(2) $(x, y, 1) \oplus \big((x', y', 1) \oplus (x'', y'', 1)\big) = \ldots = (x + x' + x'', y + y' + y'', 1) = \ldots \big((x, y, 1) \oplus (x', y', 1)\big) \oplus (x'', y'', 1)$ [I am omitting boring calculations because this is taking too long to type, sorry.]

(3) There is to be a "zero vector" in V; $\mathbf{z} = (0, 0, 1)$ does in fact have the desired property: $(x, y, 1) \oplus (0, 0, 1) = (x, y, 1) = (0, 0, 1) \oplus (x, y, 1)$.

(4) For every vector $v = (x, y, 1)$ let "$-v$" be $(-x, -y, 1) \in V$; then $v \oplus$ "$-v$" $= (0, 0, 1) = \mathbf{z}$, as needed.

(5) $a \odot (b \odot (x, y, 1)) = a \odot (bx, by, 1) = (a(bx), a(by), 1) = ((ab)x, (ab)y, 1) = (ab) \odot (x, y, 1)$

(6),(7),(8) are similar.

COMMENTS: Yes, you should know what the 8 axioms say! I don't care the order in which you list them, but please know them.

On a superficial level, some of you messed up because you can't even remember what kinds of objects your letters stand for. Above I used a, b, x, y, x' etc. to represent numbers, and u, v and $\mathbf{z}$ to represent vectors. It's quite common to let x be the name of a vector, for example, but please don't confuse yourself! $u \oplus v$ makes sense, $x \oplus y$ does not.

On a deeper level, most of you found this problem to be very confusing, and with good reason. This V is a vector space. Also, V is clearly a subset of $\mathbf{R}^4$. And $\mathbf{R}^4$ is a vector space too. But V is *not* a *subspace* of $\mathbf{R}^4$. What allows this conundrum is that we use different binary operations in the two settings. What do you get if you add $(3, 2, 1)$ to $(5, 0, 1)$? Well, these are both elements of $\mathbf{R}^4$ and, using the addition operation of the vector space called $\mathbf{R}^4$, we get $(3, 2, 1) + (5, 0, 1) = (8, 2, 2)$. But these are also elements of V, and, using the addition operation of the vector space called V, we get $(3, 2, 1) \oplus (5, 0, 1) = (8, 2, 1)$. Not the same result! Likewise, saying "The zero vector isn't in V" is both true and false: the zero vector of the vector space $\mathbf{R}^4$ is not an element of V, but the zero vector of the vector space V is indeed in there.

So remember, whenever someone wants to introduce you to a vector space, make absolutely sure you know what the elements of the set are, *and* how you are supposed to add them, *and* how you are supposed to multiply them by constants. When in doubt, you are always welcome to ask the person who is doing the introductions!

I do apologize for the length of this problem: written out carefully it takes up a lot of space and I'm sure it would have consumed a lot of your limited amount of time just to write it all down. Kudos to those who managed a fairly complete answer!

6. Prove that

$$W = \{(x, y, z, w) \mid x = 0 \text{ and } y = z\}$$

is a subspace of $\mathbf{R}^4$.

ANSWER: If $v_1 = (x_1, y_1, z_1, w_1)$ and $v_2 = (x_2, y_2, z_2, w_2)$ are elements of W and $c \in \mathbf{R}$, then by definition of W, $x_1 = x_2 = 0$, and $y_1 = z_1$ and $y_2 = z_2$. Then $v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2)$ is also in W since $x_1 + x_2 = 0$ and $y_1 + y_2 = z_1 + z_2$. Similarly $cv_1 = (cx_1, cy_1, cz_1, cw_1)$ lies in W because $cx_1 = 0$ and $cy_1 = cz_1$. Therefore, W is closed under addition and closed under scalar multiplication. Since W is clearly not empty (e.g. $(0, 0, 0, 0)$ meets the entrance requirements), it follows that W is a subspace of $\mathbf{R}^4$.

COMMENTS: You guys liked this problem a lot better than problem 5!

You do *not* need to verify that W satisfies the 8 axioms. First of all, the definition of "subspace" does not mention them! But more importantly, it is easy to prove that a subspace of a vector space is another vector space (i.e. it satisfies all 8 axioms); you just keep saying "Blah blah (e.g. the associative property) holds for all vectors in W because the equation in question holds for all vectors in V, and W is contained in V."

Just to say it again: the difference between this situation and that of problem 5 is this: every subspace of a vector space is a vector space in its own right, *re-using the very same addition and scalar multiplication operations*; a subset that's not a subspace can also be a vector space in its own right *but only if it is given its own, brand-new binary operations.*

7. Suppose $\{v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors in $\mathbf{R}^n$, and suppose A is an $n \times n$ matrix. Show that the set of vectors $\{Av_1, Av_2, \ldots, Av_n\}$ is also linearly dependent.

ANSWER 1: If the vectors v_i are linearly dependent, then there are numbers $c_1, c_2 \ldots$, not all of them equal to zero, which make $c_1 v_1 + cv_2 + \ldots c_n v_n = 0$. Multiply both sides of this equation (on the left) by A, and use the fact that for any scalar c and any vector v we have $A(cv) = c(Av)$ to get $c_1(Av_1) + c(Av_2) + \ldots c_n(Av_n) = 0$. Since not all the c_i are zero, this exhibits a nontrivial linear relation among the vectors $Av_1, \ldots Av_n$, proving that those vectors are linearly dependent.

ANSWER 2: If the vectors v_i are linearly dependent, then at least one of them can be written as a linear combination of the others. Put such a vector at the start of the list (that is, call it v_1 instead of v_{27} or whatever) so we have an equation

$$v_1 = c_2 v_2 + c_3 v_3 + \ldots c_n v_n$$

for some real numbers c_i. Multiply both sides of this equation (on the left) by A, and use the fact that for any scalar c and any vector v we have $A(cv) = c(Av)$ to get

$$(Av_1) = c_2(Av_2) + c_3(Av_3) + \ldots c_n(Av_n)$$

That is, the first of the vectors Av_i is a linear combination of the others, showing that the set of all of them is linearly dependent.

COMMENTS: You had just had a very similar problem in homework. Actually the interesting statement is the converse of this problem, which is only true when A is invertible. (The answers above show that in *our* problem, it doesn't matter whether A is invertible or not.)

I posed this problem because people have a lot of trouble with the verbiage that surrounds linear dependence/independence. Some of you made it even harder by trying to negate these things (like the negation of "all of them are zero" is not the same as "all of them are not zero"!).

I think it might have helped you a lot to stare at an example. Consider, say, the vectors $\{[1, 0, -1], [0, -1, 1], [-1, 1, 0]\}$ that we looked at in class. They are linearly dependent. Multiply them by your favorite matrix. Can you see why the product vectors Av_1, Av_2, Av_3 you computed are also linearly dependent?

Some of you presented answers involving determinants or row reduction. Yeah, meh. I don't recommend that. True, it's very useful in applications to know how to test a bunch of vectors for linear (in)dependence. But those tools you know are limited to vectors in $\mathbf{R}^n$, but there are lots of other vector spaces you might confront. Also those tools make it really hard to construct proofs. It's much easier to write the proofs — and in my opinion, much easier to understand what's really going on — if you learn to manipulate the actual definitions, and the theorems that follow from them. And, as a bonus, you'll find you don't have to deal with so many tiny numbers and subscripts and lot of "…"s.

Also, you might make a answer to this question using determinants in this case, but only because I by accident made the number of vectors v_i match the dimension of the vector space R^n; the problem would have worked just fine without that restriction, but determinants would have been useless because you'd have had non-square matrices.

8. Find a basis for $\mathcal{M}_{2,2}$ that includes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

or explain why no such basis exists.

ANSWER: Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Call these matrices M_1, M_2, M_3, M_4; obviously M_1 and M_2 are the matrices that are supposed to be in there. I need only show that $\mathcal{B}$ is a basis.

To see it spans, I will show every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear combination $\sum x_i M_i$ for some numbers x_1, x_2, x_3, x_4. Comparing the four entries of the matrices we see we need the x_i to satisfy

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ x_1 - x_2 + x_4 &= b \\ x_1 - x_2 &= c \\ x_1 + x_2 &= d \end{aligned}$$

You can row-reduce the left sides to the identity matrix, which shows that a solution exists (and is unique), or you can actually compute the solution:

$$x_1 = \frac{c+d}{2},\ x_2 = \frac{d-c}{2},\ x_3 = a - d,\ x_4 = b - c$$

Similarly to see $\mathcal{B}$ is linearly independent, observe that when $a = b = c = d = 0$, the *only* solution is $x_1 = x_2 = x_3 = x_4 = 0$.

COMMENTS: Other bases are possible, of course.

You can stop your proof after proving that $\mathcal{B}$ spans if you just observe that $\mathcal{B}$ has the right number of elements ($\mathcal{M}_{2,2}$ is easily seen to have dimension=4).

The four numbers x_i that I computed for you are what in section 4.7 are called *the coordinates of* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with respect to the basis* $\mathcal{B}$, denoted $[M]_{\mathcal{B}}$.

Now, you may ask how I *found* that basis. There are a couple of ways to do that. I started with just $\{M_1, M_2\}$, since both those matrices are supposed to be in the basis. Well, is this already a basis? A set of two elements is easily checked for linear independence: the only way a pair of elements can be linearly dependent is if one of them is a scalar multiple of the other, and that's clearly not true here. Yay! So does the set also span? Certainly not — you can see that any linear combination of the two will have equal elements on the main diagonal.

So enlarge the set: pick any third matrix M_3 that's not in the span of the first two. (My M_3 has unequal entries on the main diagonal, so that's a win.) When you enlarge your set this way, you will never destroy the linear independence. (That's a theorem.) And perhaps now this set spans. Try to prove it as I did with my basis and ... it fails. You find some combinations of a, b, c, d which are not in the span. Very well, once again: take something that's not already in the span (like my M_4) and toss it in; the enlarged set will still be linearly independent, and now, finally, we find that it also spans.

Some students took a different approach. Rather than building up a linearly independent set until it spans, you can start with a spanning set and then toss out redundant vectors until what's left is linearly independent. A natural starting point is to combine M_1 and M_2 with the four matrices that we always take as the standard basis for $\mathcal{M}_{\in,\in}$. You have lots of choices for what to toss out. There's a row-reduction method described in the text that will tell you what your options are. Toss out the redundant ones and what's left will make a basis.

Overall I will say the test was long and you guys had a hard time finishing it as well as you would like. I'm sorry about that. This is why I use the mean-and-standard-deviation grading scale. The standard deviation was large enough that no one failed but, frankly, some of you should probably have a chat with me about how you're doing. We have two weeks left and will discover a lot of interesting stuff in Chapter 5 but it's going to be a fair amount of work and will require you to be comfortable with what we have done throughout the semester. Take advantage of the remaining time so that you can be prepared for the final. I will help you but you have to take the initiative. Good luck!