I thought you might appreciate better the proof that $0 \cdot x = 0$ for every real x, if I showed you an example where it isn't true!

This example is based on the following observation: when you have two numbers x and y which are both close to 1, you can approximate their product very easily: if they're both close to 1 we can write them as $x = 1 + a$ and $y = 1 + b$ for some small numbers a and b. Then $xy = (1 + a)(1 + b) = 1 + a + b + ab$, but if a and b are both already small then their product ab is tiny, i.e. $xy \approx 1 + a + b = 1 + (x - 1) + (y - 1) = x + y - 1$. In the following example we will elevate this approximation to the status of a definition!

So, suppose I asked you to bring me the real numbers. You know you need a set $\mathbf{R}$ and two binary operations on it called **PLUS** and **TIMES**. They should have identity elements ("neutral elements") which we will call **ZERO** and **ONE** respectively. Keeping in mind the previous paragraph, you might decide to create your own private "real numbers" as follows:

(1) $\mathbf{R}$ = the same *set* of real numbers as everyone else uses; and
(2) **PLUS** = ordinary addition: (x **PLUS** y) $= x + y$; but
(3) **TIMES** is defined by (x **TIMES** y) $= x + y - 1$

(For example, 0.9 **TIMES** $1.2 = 0.9 + 1.2 - 1 = 1.1$ instead of $0.9 \cdot 1.2 = 1.08$.)

The axioms for **PLUS** are obviously satisfied; the identity element **ZERO** is 0, and the additive inverse of each x is $-x$. But the axioms for **TIMES** are satisfied, too! For example, let's check the commutative property: x **PLUS** $y = (x + y) - 1 = (y + x) - 1 = y$ **PLUS** x. The identity element **ONE** turns out to be 1, since for any x, x **TIMES ONE** $= x +$ **ONE** $- 1$ will be equal to x if and only if **ONE** $= 1$. The multiplicative inverse of each x is $2 - x$, since x **TIMES** $(2 - x) = x + (2 - x) - 1 = 1$.

So life is good in this new set of "real numbers"; the eight axioms are all satisfied. But then we notice that **ZERO TIMES** x = **ZERO** $+ x - 1 = 0 + x - 1 = x - 1$ is not equal to **ZERO** (except when $x = 1$).

The source of the problem is that the distributive property does not hold: that property would insist that a **TIMES** (b **PLUS** c) = (a **TIMES** b) **PLUS** (a **TIMES** c) but instead, a **TIMES** (b **PLUS** c) is equal to

$$a \textbf{ TIMES } (b + c) = a + (b + c) - 1 = (a + b - 1) + c = (a \textbf{ TIMES } b) \textbf{ PLUS } c$$

Part of that matches what we wanted to get, but instead of $\ldots + c$ we were expecting to get $\ldots + a$ **TIMES** c, which equals $a + c - 1 = c + (a - 1)$, so the hoped-for equality does not hold (except for $a = 1$).

This example should not be terribly surprising: the claim $0 \cdot x = 0$ implies some connection between multiplication and addition (since 0 is the *additive* identity elements), and without the distributive property there need not be *any* connection between the two operations.