The set F is a subset of the set of real numbers, where all the axioms for a field (the commutative and associative laws, etc.) are already satisfied. We simply need to observe that the sum and product of any two elements of F will again be elements of F; then F will satisfy most of the axioms of a field just because $\mathbf{R}$ does. (We say that F *inherits* these properties from $\mathbf{R}$.) We do also have to check that the special elements 0 and 1 lie in F (and they do). The fact that $\mathbf{R}$ is a field means that every element of F has additive and multiplcative inverses somewhere in $\mathbf{R}$; we just need to check that those inverses lie in F. Again, it's trivial that for every $z \in F$, the real number $-z$ is also in F, so we have only one thing left: to decide whether the multiplicative inverse of every $z \neq 0$ in F also lies in F.

When $a = \pi$ the answer is 'no'. Already $z = \pi$ itself is a nonzero element of F and so it has an inverse $1/\pi$ in $\mathbf{R}$, but this inverse is NOT in F. That is, $1/\pi$ cannot be written as a polynomial in π with rational coefficients. Indeed, if $1/\pi = c_0 + c_1\pi + c_2\pi^2 + \dots c_n\pi^n$ then π would be a root of the polynomial

$$c_n X^{n+1} + c_{n-1} X^n + \dots c_1 X^2 + c_0 X - 1$$

We could then clear denominators from the rational coefficients and get an integer polynomial whose root is π. That would make π be an algebraic number, which I assured you in class that it is not.

When $a = 5^{1/3}$, however, the answer is yes, and you can actually compute the inverse of any element of F, but really it's quicker (and more impressive!) to do this without much calculation. Indeed all you need is this fact:

F is a finite-dimensional vector space over $\mathbf{Q}$

(That means the axioms for a vector space are satisfied, where "scalar" means "element of $\mathbf{Q}$" rather than "element of $\mathbf{R}$", which is the convention in Math 341.) I will leave it to you to think about those axioms and see F satisfies them. When $a = 5^{1/3}$, the finite-dimensionality of F comes from observing that F is spanned by $\{1, a, a^2\}$. (It's actually true that this set is also linearly independent, although that takes just a bit of work to actually prove, and we don't need to know it!)

Now here is the trick: if z is any single element of F, we may define a function $L_z : F \to F$ by the recipe $L_z(y) = zy$ for every element $y \in F$. (Make sure you understand this. For example, what is the corresponding function $L_i : \mathbf{C} \to \mathbf{C}$?). For each $z \in F$, this L_z is actually a *linear transformation*, since for every $y_1 \in F$, $y_2 \in F$, and $c \in \mathbf{Q}$ we have

$$L_z(y_1 + y_2) = L_z(y_1) + L_z(y_2) \qquad \text{and} \qquad L_z(cy_1) = cL_z(y_1)$$

simply because the distributive, associative, and commutative laws hold in $\mathbf{R}$.

Now if $z = 0$ then certainly L_z is the zero map and its kernel is all of F. But for any other $z \in F$, I claim $\ker(L_z) = \{0\}$. Indeed, if $L_z(y) = 0$, that would mean $zy = 0$, and since z is a nonzero real number, this forces $y = 0$.

But you had an important theorem in Linear Algebra: the condition $\ker(L_z) = \{0\}$ shows L_z to be an *invertible* linear map. (Perhaps you remember it this way: a square matrix with kernel=$\{0\}$ has an inverse.) In particular, L_z is also onto, that is, for every $w \in F$ the equation $L_z(y) = w$ has a solution for some $y \in F$. If we apply this in particular to the element $w = 1$ in F, we see that the equation $zy = 1$ has a solution for some $y \in F$, i.e. z has an inverse in F.

So we are done, and the only thing we used about a is that F is finite-dimensional over $\mathbf{Q}$. (If you want to make the proof constructive, and actually *compute* the inverse of an element $z \in F$, you could use the basis above to turn everything into some calculations with 3×3 matrices. The proof then hinges on whether the denominator involved, which is $\det(L_z)$, is nonzero. That denominator turns out to be $A^3 + 5B^3 + 25C^3 - 15ABC$ when $z = A + Ba + Ca^2$; you can show that there are no rational triples (A, B, C) for which this vanishes, except $(A, B, C) = (0, 0, 0)$, by considering the powers of 5 involved in A, B, and C.)

If you happen to know some Algebraic Number Theory, you can also compute the inverse of any nonzero element of F by noting $z \cdot z' \cdot z'' = N(z)$ where z' and z'' are the complex numbers obtained from z by replacing a by the other two complex cube roots of 5, and $N(z)$ is the same rational number that shows up as the determinant of L_z. It's not hard to show that z' and z'' must be nonzero as well, which makes $N(z)$ the product of three nonzero complex numbers and hence itself is nonzero, allowing us to divide by $N(z)$ to get $z^{-1} = z'z''/N(z)$.