M365C (Rusin) HW4 comments

1. Plenty of examples exist, e.g. $A = \mathbf{R} - \{0\}$ and $B = \{0\}$.

2. *All* subsets A of any discrete metric space X are open! If $a \in A$ then the ball $N_1(a)$ of radius 1 is completely contained inside A since that ball consists of nothing but $\{a\}$!

3. Since the open sets are precisely the unions of open balls, it suffices to show that every open ball (in either metric) is open in the other metric. (When two metrics have this relationship, we say the metrics are *equivalent*.)

In fact we will show that the two metrics are what we call *uniformly equivalent*: we will show that there exist constants C and D such that for all points P and Q in the plane,

$$d^E(P,Q) \le C\, d^T(P,Q) \qquad \text{and} \qquad d^T(P,Q) \le D\, d^E(P,Q)$$

In fact this statement is true when $C = 1$ and $D = 2$. If we name the coordinates of these points $P = (a, b)$ and $Q = (c, d)$, then draw the triangle with vertices P, Q, and $R = (a, d)$. This triangle has legs of length $|a - c|$ and $|b - d|$ and a hypotenuse of length $d^E(P,Q)$. Surely each leg is shorter than the hypotenuse; adding those equations together shows $d^T(P,Q) \le 2\, d^E(P,Q)$. On the other hand the triangle inequality in $\mathbf{R}^2$ (applied to the three points P, Q, R) shows that $d^E(P,Q) \le d^T(P,Q)$. This completes the proof of the inequalities above.

Now, from these inequalities we prove the metrics are equivalent, that is, I will show every T-open ball is E-open and vice versa. The arguments are completely symmetric so I will only do one of them. Let $B = N_r(P)$ be a T-open ball and suppose Q is a point inside it. I have to show that I can find an E-ball around Q that is completely contained inside B. To do this, let $s = d^T(P,Q)$; then $s < r$ because $Q \in B$. I claim that the E-ball of radius $(r - s)/D$ lies completely inside B. Indeed, if R is any point in this new ball, then $d^E(R,Q) < (r - s)/D$. From the inequalities of the previous paragraph, $d^T(R,Q) < (r - s)$. Then from the triangle inequality (which metric T is supposed to saitisfy!) $d^T(P,R) < s + (r - s) = r$, i.e. $R \in B$. Since this is true for every R, the E-ball of radius $(r - s)/D$ around Q does indeed lie completely inside B. And since that statement is true for every $Q \in B$, the set B is E-open. To say that again: every T-ball is E-open. By taking unions of these T-balls, we conclude that every T-open set is E-open. The opposite inequality (the one with C in it) shows in the exact same way that every E-open ball is T-open.

4. (a) Use the triangle inequality with x, y, and each $a \in A$; take the inf over all such a.

For (b), note that if $x \in A$ then surely $d(x, A) \le d(x, x) = 0$. And also if x is a limit point of A, then by definition there are elements of A in the balls $N_{1/n}(x)$ around x, so $d(x, A) \le 1/n$ for every n, and hence again $d(x, A) = 0$.

Conversely if $d(x, A)$ we can prove x is in the closure of A. If x is in A we are already done and if not then use the definition of the infimum to know that for every $r > 0$ there is an $a \in A$ with $d(x, a) < r$; this a is different from x, so $A \cap N_r(x)$ is neither empty nor the singleton $\{x\}$, which by definition makes x a limit point of A, and hence it lies in $\bar{A}$.

5. You may simply define $U = \{x \in X \mid d(x, A) < d(x, B)\}$ and similarly for B. Then $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \emptyset$. You need only show U and V are open. But if $x \in U$, let $r = d(x, B) - d(x, A) > 0$; I claim the ball $N_{r/3}(x) \subseteq U$. For if y is an element of this ball, then $d(x, y) < r/3$, which by problem 4a means $d(y, A) \leq d(x, A) + r/3$ and similarly $d(x, B) \leq d(y, B) + r/3$. Add and rearrange terms to see

$$d(y, B) - d(y, A) \geq d(x, B) - d(x, A) - 2r/3 = r/3 > 0$$

so $y \in U$ as well.

6. I already gave hints when I posed the problem so let me focus on how you might write up your solution. (The write-up below is rather pedantic. There are other ways to work through the same hints, giving proofs that sound really different, but they are using the same ideas. Feel free to use a language and style that works best for you!)

There are several large sets here: the set A of points, the set Γ of subscripts, the set S of open sets whose union includes all of A, and in a moment we will also discuss the set R of open intervals whose two endpoints are both rational numbers. We will get the elements in these sets to correspond to elements in the other sets. For starters, we have the obvious mapping from Γ to S , matching γ with O_γ. This pairing is onto, but not necessarily one-to-one. (There is no reason we can't have some of the O_γs repeated!)

The fact that the open sets in S covers A means that for each $a \in A$ there is an open set $O \in S$ which includes it. This allows us to define a function $f : A \to \Gamma$ so that $a \in O_{f(a)}$. Now, the fact that this set is open and a is in it means, by definition, that there is an open interval (an open ball in $\mathbf{R}$) around a which is contained in $O_{f(a)}$; write that interval as (b, c), say, so $b < a < c$. Find rational numbers d between a and b, and e between a and c; then $I = (d, e)$ is an interval with three important properties: It's in R, it contains a, and it's contained in $O_{f(a)}$. If we call the interval $g(a)$, then we have defined a function $g : A \to R$.

Now, R is a countable set: for each rational number d there are only countably many intervals $(d, e) \in R$ since $\mathbf{Q}$ is countable. Thus R is a countable union of countable sets, hence countable. It follows that the image $R' = g(A)$ (a subset of R) is also countable. For each interval $I \in g(A)$, pick an element $h(I) \in A$ with $g(h(I)) = I$ (that is, h is a "right inverse" of g). Observe that this will identify a countable subset of $A' = h(R')$ of A, which has the feature that for *every* $a \in A$, there is an $a' \in A'$ with $g(a) = g(a')$.

Finally: let $\Gamma' = f(A')$. This is a subset of Γ, and it's countable because A' is. So this will be a countable subcover of our original cover, as soon as we prove that it does, in fact, cover A.

But this is not hard: for any $a \in A$ we know there is an $a' \in A'$ with $g(a) = g(a')$. Then recall that the point a is contained in the interval $g(a)$, which means $a \in g(a')$. On the other hand, we noted that $g(a')$ in turn is contained in $O_{f(a')}$, and $f(a') \in \Gamma'$. So a is indeed covered by the open set $O_{f(a')}$ of our subcover.