M365C (Rusin) HW6 comments

1. If $|x| \geq 1$ then the individual terms do not approach zero, so the series cannot converge. If $0 \leq x < 1$ then the series converges by (direct) comparison to the geometric series $\sum x^n$. This in turn shows the series converges-absolutely for $-1 < x < 0$, and we proved in class that this implies convergence.

2. Precisely the question is whether the partial sums get closer to the element $L = 2/(2-x)$ of $C^0[0,1]$ that purports to be the limit. So let us note that the partial sums $S_N = \sum_{n=0}^{N}(2^{-n})x^n$ may be expressed in closed form as

$$\frac{1-(x/2)^{N+1}}{1-(x/2)}$$

Thus the distance $d(S_n, L) = \int_0^1 |L - S_n|\, dx$ is the integral of $(x/2)^{N+1}/(1-(x/2))$ over this interval. The denominator takes values between $1/2$ and 1, so the integrand is at most $2(x/2)^{N+1}$, and thus the integral is at most $1/2^N(N+2)$, which clearly decreases to 0 as $N \to \infty$,

There's a lot you can say about that last sentence in question 2.

The series $\sum x^n$ converges pointwise to $F = 1/(1-x)$, but only on $[0,1)$. One might ask whether this series converges to F in the sense of the previous paragraph, but that's actually meaningless: F is not even in this metric space, so there is no reason we should expect the distances to the S_n to behave well. There are larger sets of function than $C^0[0,1]$ on which we may use the same integral formula to compute distances, but they don't include our F; after all, how far is F from, say, the constant function 0? The intergral does not yield a real number for this distance.

Well, perhaps the series converges to some element $G \in C^e[0,1]$ other than F? That's impossible too: the sequence of partial sums would have to converge to G in the metric space sense, which would in particular mean that they would form a Cauchy Sequence. But these partial sums don't grow close together: we compute the distance from S_N to S_M by integrating $|S_N - S_M| = |x^{N+1} - x^{M+1}|/(1-x)$. This rational function may be expressed as (assuming $M < N$, say)

$$x^{M+1}(1-x^{N-M})/(1-x) = x^{M+1}(1+x+x^2+\ldots+x^{N-M-1}) = \sum_{i=M+1}^{N} x^i$$

whose integral from 0 to 1 is $\sum_{i=M+1}^{N}(1/i)$. But those numbers are precisely the distances (in $\mathbf{R}$) between the partial sums of the harmonic series, and we know they do not form a Cauchy sequence because the harmonic series does not converge. (You could also give estimates that show, for example, that $d(S_N, S_{2N})$ is approximately $\ln(2)$ for all N, so that in particular there is no way to force these distances to get small simply by making N large.)

3. For the square-root function, you can simply use $\delta = \epsilon^2$; then in this case it's clear that whenever $d(x,0) < \delta$, it follows that $d(f(x), f(0)) < \epsilon$.

The reciprocal function is a little trickier. At a point $a \neq 0$, we can calculate that $d(f(x), f(a)) = |x-a|/|xa|$, which is small if the numerator is small. We will be computing this distance for x that are close to a, so we expect the numerator to be close to a^2, so we are tempted to use $\delta = a^2\epsilon$; then $|x-a| < \delta$ *should approximately mean* that $|1/x - 1/a| < \epsilon$, as desired. Things don't seem to quite work when $|x| < |a|$ so we might try to make the numerator $|x - a|$ a little smaller, say by taking $\delta = (1/2)a^2\epsilon$. In that case, $|x - a| < \delta$ implies that $|1/x - 1/a| < K\epsilon$, (where now K works out to be $|a|/2|x|$). That would be just what we want if we knew that $K \leq 1$, but *that* will only be true if $|x| > |a|/2$, and there doesn't seem to be any way to get that inequality. But on the other hand, — there is! We will surely have $|x| > |a|/2$ if x is close to a, right? I mean, to violate this inequality, we would have to have x far from a, being a distance of at least $|a| - |a|/2 = |a|/2$ away from a. So let's prevent this. I will show that this δ works:

$$\text{For each } a \neq 0 \text{ and each } \epsilon > 0 \text{ let } \delta = \min(|a|/2, a^2\epsilon/2).$$

Let's see why this works. Clearly I have specified a positive number δ. I have to show that for each x with $d(x, a) < \delta$, something nice happens. Keep in mind that by construction, $\delta \leq |a|/2$ and $\delta \leq a^2\epsilon$ so $d(x, a) < \delta$ implies two things: $|x-a| < |a|/2$ and $|x-a| < a^2\epsilon/2$. The first of these, together with the Triangle Inequality, implies that $|x| = |x - 0| \geq |a - 0| - |x - a| > |a|/2$. Then as before we compute

$$d(f(x) - f(a)) = |1/x - 1/a| = |x - a|/|x||a| < \delta/|x||a| < \delta/(|a|^2/2)$$

but since $\delta \leq a^2\epsilon/2$, this last quantity is at most ϵ. That is: the "something nice" does in fact happen. Therefore, $f(x) = 1/x$ is indeed continuous at a.

Some of you might prefer this approach: Let's assume for simplicity that $a > 0$; the case $a < 0$ is similar. Given ϵ, we'd like to find a range of xs where $1/a - \epsilon < 1/x < 1/a + \epsilon$. Well, as long as x itself is also positive, these inequalities are equivalent to

$$\frac{1}{1/a - \epsilon} < x < \frac{1}{1/a - \epsilon} = \frac{1}{\left(\frac{1-a\epsilon}{a}\right)} = \frac{a}{1 - a\epsilon} = a + \frac{a^2\epsilon}{1 - a\epsilon}$$

where I have simplified one endpoint for you but the other is similarly expressible as $a - \frac{a^2\epsilon}{1+a\epsilon}$. In other words, the desired inequality $(d(f(x), f(a)) < \epsilon)$ will hold as long as x is also positive and does not go too far from a: no further than $\frac{a^2\epsilon}{1-a\epsilon}$ to the right and no further than $\frac{a^2\epsilon}{1+a\epsilon}$ to the left. (And x will indeed be positive as long as we go no further than a distance of a to the left!) Therefore, a number x will give us our desired inequality as long as its distance from a is less than

$$\delta = \min(a, \frac{a^2\epsilon}{1 - a\epsilon}, \frac{a^2\epsilon}{1 + a\epsilon})$$

(It's not hard to show that the middle one of these three is never the minimum, but technically speaking we are not required to make that observation! We should, however, note that our calculations assume that $\epsilon < 1/a$ because otherwise the middle term would be negative, and some of our manipulations with inequalities would be invalid. But I leave it to you to puzzle out the logic of this statement: we can prove continuity as long as we can find a δ only for all *small* ϵ.)

4. Every function *from* a discrete space is continuous everywhere. You may use the definition of continuity at a point: no matter what ϵ is, use $\delta = 1$, because then $d(x, a) < \delta$ implies $x = a$, so that $d(f(x), f(a)) = 0 < \epsilon$.

Functions from $\mathbf{R}$ *into* a discrete space are never continuous unless they are constant. If such an f is continuous at a point $a \in \mathbf{R}$, then apply the definition of continuity with $\epsilon = 1$: there should exist a δ for which $|x - a| < \delta$ makes $d(f(x), f(a)) < 1$, but in a discrete space, the only number less than 1 that can be a distance is 0; that is, $d(f(x), f(a)) < 1$ implies $d(f(x), f(a)) = 0$, which in turn implies that $f(x) = f(a)$. In other words, continuity at a implies there is an interval $(a - \delta, a + \delta)$ around a where f stays constant. Of course the same reasoning applies to any other $b \in \mathbf{R}$: f must stay constant near b, too.

It's just a little tricky to turn this idea of being "locally constant" into a proof that f is actually constant on the whole real line. Let c be any point to the right (say) of a; I will show $f(c) = f(a)$. Consider the set

$$X = \{x \in \mathbf{R} \,|\, x \leq c \text{ and } f(x) = f(a)\}$$

This set is bounded and not empty, so let $b = \sup(X) \leq c$. As noted above, whatever value f has at b will also be the value of f at all points in a neighborhood of c. But every such neighborhood has to meet points of X, where $f(x) = f(a)$. So we must have $f(b) = f(a)$. Now, if b is different from c, then $b < c$ so b is in X; but then is an interval on either side of b which is contained inside X too, which violates the fact that $b = \sup(X)$. So we conclude that $b = c$, so that $f(c) = f(b) = f(a)$, The same works for points to the left of a, so that f is constant all the way across the real line.

These problems are solved quite readily using the idea that a function is continuous (on its whole domain) iff the preimage of every open set (in the codomain) is open (in the domain). Simply note that in a discrete space, every subset is open, including every singleton.

The constancy of continuous functions into a discrete space then depends on the fact that $\mathbf{R}$ cannot be decomposed as a disjoint union of 2 open subsets. Spaces with this property are called *connected*. This condition is related to, but not quite the same as, "being able to draw the whole thing without lifting up your pencil".

5. The classic "counterexample" is $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

which is obviously not continuous. It's not really a counterexample because the interval $A = (-\infty, 0)$ is not closed, but it serves as a testing ground to discover the nuances in the proof that I will give.

So suppose f is continous on both A and B separately. (More precisely, the restrictions $f|_A : A \to Y$ and $f|_B : B \to Y$ are both continuous.) Let us prove that f is continous at each point $x \in X$. There are no points $x \in X$ that are in neither A nor B, by hypothesis, so we can consider the cases $x \in A \cap B$ and $x \in A \cap B^c$ separately; the remaining case $x \in A^c \cap B$ is obviously the same as the previous one.

If $x \in A \cap B$, we simply use the definition of continuity on $f|_A$ and $f|_B$ separately: given an $\epsilon > 0$ we know there exist positive numbers δ_A and δ_B such that

$$\begin{array}{ccccc} x' \in A & \text{and} & d_X(x', x) < \delta_A & \Rightarrow & d_Y(f(x'), f(x)) < \epsilon \\ x' \in B & \text{and} & d_X(x', x) < \delta_B & \Rightarrow & d_Y(f(x'), f(x)) < \epsilon \end{array}$$

Simply take $\delta = \min(\delta_A . \delta_B)$. Then for any $x' \in X$, we know that $d_X(x', x) < \delta \Rightarrow d_Y(f(x'), f(x)) < \epsilon$, whether $x' \in A$ or $x' \in B$ (or both).

If instead $x \in A \cap B^c$, we first use the definition of continuity on $f|_A$ to find a $\delta_A > 0$ such that

$$x' \in A \qquad \text{and} \qquad d_X(x', x) < \delta_A \qquad \Rightarrow \qquad d_Y(f(x'), f(x)) < \epsilon$$

Then use the fact that B is closed, i.e. B^c is open, to find a δ_B such that $N_{\delta_B}(x) \subseteq B^c$, that is,

$$x' \in X \qquad \text{and} \qquad d_X(x', x) < \delta_B \qquad \Rightarrow \qquad x' \in B^c$$

Again define $\delta = \min(\delta_A, \delta_B)$. Then for any $x' \in X$, we know that $d_X(x', x) < \delta$ implies first that $x' \in B^c$, which means $x' \in A$ since $X = A \cup B$. And then we can use the definition of δ_A to deduce $d_Y(f(x'), f(x)) < \epsilon$, as desired.

Again, this result is true in the topological setting too (i.e. using only open and closed sets rather than ϵ and δ).