M365C (Rusin) HW7 – comments

1. You need to find a disk around each point $P = (a, b) \in \mathbf{R}^2$ that f carries into the interval $(C - \epsilon, C + \epsilon)$, where $C = f(P) = a + b$, that is, you need to know (a, b) is in the interior of the pre-image of this interval. Since the preimage of each value $c \in \mathbf{R}$ is the line $x + y = c$ of slope -1, the preimage of the interval is a stripe of slope -1 (i.e. a union of these lines), and it's pretty clear from a picture that there is a disk centered around (a, b) that lies inside there.

In formulas, if we are given an $\epsilon > 0$, let $\delta = \epsilon/2$. Then for any point $Q = (x, y) \in \mathbf{R}^2$ with $d(P, Q) < \delta$, we know $|x - a| < \delta$ and $|y - b| < \delta$ since each of these legs of a triangle is shorter than the hypotenuse of length $d(P, Q)$. Then $a - \delta < x < a + \delta$ and likewise for y, so $f(Q) = x + y$ lies between $f(P) + 2\delta$ and $f(P) - 2\delta$, i.e. $|f(Q) - f(P)| < \epsilon$, as desired.

With a bit more work, we can show that the larger value $\delta = \epsilon/\sqrt{2}$ also works, but why bother? The point is simply to show that a δ exists; there is no virtue in finding the *biggest possible* disk around P that lies inside the preimage of $(C - \epsilon, C + \epsilon)$.

You might want to observe that the δ that I proposed does not depend on the point P, so that in fact addition is *uniformly* continuous. What about multiplication?

2. The supremum distance between f_n and the zero function f_0 is n, so obviously these distances are not tending to zero: the sequence $\{f_n\}$ does not converge to f_0 in this metric.

For any positive real p, $d_p(f_n, f_0)^p = \int_0^{1/n} |n(1 - nx)|^p \, dx$ may be computed by substitution: let $u = 1 - nx$ to get $\int_0^1 n^{p-1} u^p \, du = n^{p-1}/(p + 1)$. So the distance from f_n to f_0 is a constant (i.e. independent of n) multiple of $n^{1-(1/p)}$. If $p = 1$, this distance is 1 for every n, and not converging to zero, and if $p > 1$, the distance is a positive power of n and hence increases without bound as n increases, so in neither of these cases does $\{f_n\}$ converge to f_0. On the other hand if $p < 1$, then the distance is a negative power of n and hence decreases to 0, meaning $f_n \to f_0$ in this case.

For each $x > 0$ the sequence of *numbers* $f_n(x)$ converges to 0; in fact all the terms in the sequence are exactly 0 as soon as $n > 1/x$. On the other hand the sequence of numbers $f_n(0) = n$ clearly diverges. Thus $\{f_n\}$ converges pointwise to f_0 on $(0, 1]$ but not on $[0, 1]$.

Note that if f and g are continuous functions and $M = \max_{x \in [0,1]} |f(x) - g(x)|$, then $d_p(f, g) = (\int_0^1 |f(x) - g(x)|^p \, dx)^{1/p} \le (\int_0^1 M^p \, dx)^{1/p} = M$. On the other hand we could use the definition of continuity to show that for every $\epsilon > 0$ there is a subinterval of some length $L = 2\delta < 1$ on which $|f(x) - g(x)| > M - \epsilon = N$, say; then integral inequalities like we just used will also show $d_p(f, g) \ge L^{1/p} N$, and since $L < 1$, this quantity approaches N as $p \to \infty$. So $\lim_{p \to \infty} d_p(f, g) \ge M - \epsilon$ for every $\epsilon > 0$, and we conclude $\lim_{p \to \infty} d_p(f, g) = M$. So it is quite natural to denote this M as $d_\infty(f, g)$.

3. For every a the evaluation function e_a is continuous at each point $f_0 \in C^0[0, 1]$, if we use the supremum metric there. Indeed this e_a is uniformly continuous; we may use simply $\delta = \epsilon$. For if $d(f, f_0) < \delta$, then $\max(|f(x) - f_0(x)|) < \epsilon$; but then $|e_a(f) - e_a(f_0)| = |f(a) - f_0(a)| \le \max(|f(x) - f_0(x)|) < \epsilon$, as desired.

But for every a we can also show that e_a is not continuous *anywhere* if we use the metric d_1: as in the previous problem we can make a L^1-convergent sequence of "bump" functions f_n with their maxima at a. Then $e_a(f_0+f_n) = e_a(f_0)+e_a(f_n)$ will diverge from $e_a(f_0)$ as $n \to \infty$ even though $\{f_0 + f_n\}$ converges to f_0 in the L^1 metric. This means e_a cannot be continuous at f_0 (for every function f_0.)

4. I intended you to use Theorem 4.4. You should begin with the observation that the identity function $I_X : X \to X$ on any metric space X is continuous (that means $I_X(x) = x$ for every $x \in X$); the proof simply uses $\delta = \epsilon$. Even easier: constant functions $c : X \to Y$ are always continuous — you can use any positive δ you like!

So in particular $I_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$ is continuous. Then the squaring function $m_2 : \mathbf{R} \to \mathbf{R}$ is the product of two continuous functions ($m_2 = I_{\mathbf{R}} \cdot I_{\mathbf{R}}$) and hence continuous. By induction we similarly prove each nth power function $m_n(x) = x^n$ is continuous at every point of $\mathbf{R}$. Then the monomial functions cx^n are products and hence continuous, and finally all polynomials, being sums of continuous monomial functions, are also continuous at every point in $\mathbf{R}$.

Then rational functions are continuous at every point in their domain: as quotients of polynomials they are continuous wherever the denominator is nonzero, and the domain is also the set of points where the denominator is nonzero.

Note that this argument does not claim that a rational function like $(x^2-1)/(x-1)$ is continuous at $x = 1$: even though we usually cavalierly claim that $(x^2-1)/(x-1) = x+1$, which *is* defined (and continuous) at $x = 1$, it's not quite the same function as $(x^2-1)/(x-1)$ because the domain is different!

5. As proved in class, f will be uniformly continuous on any compact interval.

It follows that it's uniformly continuous on every bounded interval I since if we find (for each $\epsilon > 0$) a positive δ that works for every point of the closure $\bar{I}$, then a fortiori it works at each point of I.

So to get an interval for part (b), you must use an unbounded interval such as $I = (0, \infty)$. And indeed this f is not uniformly continuous there. It's not sufficient to say "I could not find δ that worked equally well for all the points of I" — you must show that no such δ exists (for some particular positive ϵ).

So let me show that no δ works for say $\epsilon = 1$. That is, I will show that there is no single $\delta > 0$ for which

$$\text{for all } x \text{ and } y \text{ in } \mathbf{R}, \qquad |y-x| < \delta \Rightarrow |y^3 - x^3| < 1$$

If such a δ existed, then in particular we could apply the stated property to $y = x + \delta/2$. But then $y^3 - x^3 = 3x^2\delta + 3x\delta^2 + \delta^3 > 3x^2\delta$, and this is not less than 1 as soon as $x^2 > 1/(3\delta)$.

Informally you should get used to this idea: for smooth functions $f : \mathbf{R} \to \mathbf{R}$ (such as the cubing function here), in order to prove continuity of f at a point a you will find a δ which is approximately $\epsilon/|f'(a)|$. So a function will not be uniformly continuous on a set unless its derivative stays bounded there. For the cubing function f we know $f'(x) = 3x^2$ is unbounded on my I.