M365C (Rusin) HW8 – some comments

1. Suppose $\{f_n\}$ converges uniformly to f. Then given ϵ, we know that there is a single N such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N$ and for every $x \in X$; in particular, $\sup(|f_n(x) - f(x)|) < \epsilon$ for these n, which is to say $d_\infty(f_n, f) < \epsilon$ for all $n > N$. Since this is true for every $\epsilon > 0$, the sequence of numbers $d_\infty(f_n, f)$ converges to 0.

Conversely if that sequence of distances converges to 0, then given $\epsilon > 0$ all but a finite number of the distances will be less than ϵ. Using the definition of this distance, that means that for every x we must have $|f_n(x) - f(x)| < \epsilon$ too, which implies that $\{f_n\}$ converges uniformly to f.

I should perhaps also have taken the opportunity to make it clear that rearranging the quantifiers from one definition to get the other shows this: a sequence of functions that converges uniformly to f also converges pointwise to f, but the converse cannot be expected to be true. Indeed the converse *isn't* true: there are sequences that converge pointwise but not uniformly; a simple example is the set of functions $f_n(x) = x^n$ on the *open* interval $(0, 1)$, which converges pointwise to the zero function, but the convergence is not uniform. (For a given ϵ, the values of N that you need in the definition will increase to infinity as $x \to 1^-$.)

On the other hand it is a theorem that when X is compact, any sequence which converges pointwise will also converge uniformly.

2. Use the fact that $f(a+h)-f(a-h) = [f(a+h)-f(a)]-[f(a-h)-f(a)]$, together with the observation that $-h$ approaches zero when h does. You will conclude $f^*(a) = 2f'(a)$.

But note that for *any* even function (i.e. $f(-x) = f(x)$), differentiable or not, we will have $f^*(0) = 0$. You might specifically use $f(x) = |x|$ or $f(x) = x^{2/3}$ to see that the existence of $f^*(a)$ does not imply $f'(a)$ exists.

3. You must use the limit-definition of derivative: $f'(0) = \lim_{h\to 0} f(h)/h$. The values of this difference quotient are all zero for $h < 0$ and for positive h are the same as $e^{-1/h}/h = z/e^z$ where $z = 1/h$ is a (large) positive number. The only fact you need to know about the exponential function (from Calculus) is that the limit of this ratio as $z \to \infty$ is zero, (You could, for example, use L'Hôpital's Rule together with the facts that the derivative of e^z is e^z and the fact that $e > 1$; the latter pieces of information make e^z into an increasing function which has no upper bound, so that $1/e^z \to 0$.)

One can show that f has derivatives of all orders at all points other than the origin. That's obvious at negative points and at positive points you can show by induction that the kth derivative at a point $x > 0$ may be expressed as $e^{-1/x}P_k(x)/x^{2k}$ for some polynomial P of degree $k - 1$. As a consequence you can also show that for every k, $f^{(k)}(0)$ exists and equals zero; by definition that amounts to showing $e^{-1/h}P_k(h)/h^{2k+1}$ tends to 0 as $h \to 0^+$. In particular, this function f *has* a Taylor series at 0, but it's simply the series $0 + 0x + 0x^2 + 0x^3 + \ldots$. This series obviously converges to $f(x)$ when $x \le 0$ but does not converge to $f(x)$ for any positive x. (We say f is a C^∞ function but not a C^ω function when the Taylor series exists but does not converge in any neighborhood of zero.)

4. We did this in class: if f is increasing then the difference quotient $(f(x+h)-f(x))/h$ is positive for every nonzero h, and hence the limit $f'(x)$ must be positive or zero (if it exists). Conversely if f' is everywhere positive then the Mean Value Theorem shows that $f(y)-f(x)=f'(c)(y-x)$ will be positive whenever $y>x$, and so f is increasing.

(I erred when stating the problem: a differentiable, increasing function need not have a *positive* derivative; we can only show each $f'(x)\geq 0$, as the function $f(x)=x^3$ shows.)

The function $f(x)=-1/x$ is defined and differentiable at all nonzero x, and $f'(x)>0$ for all such x, but f is not increasing (on its whole domain): obviously $1>-1$ but $f(1)=-1$ is not greater than $f(-1)=+1$. (So be careful when stating where a function is increasing: it can be increasing on sets A and B but not increasing on $A\cup B$!)

5. Given any $\epsilon>0$ let $\delta=\epsilon/5$. Then whenever $|x-y|<\delta$ we have $|f(x)-f(y)|=|f'(c)\,(x-y)|$ for some c between x and y, which is at most $|f'(c)|\cdot\delta<5\cdot(\epsilon/5)=\epsilon$. So f is uniformly continuous on its domain. (Of course there is nothing special about the number 5; any bound on the size of the derivative will do.)