M365C (Rusin) HW9 – some comments

1. Suppose $\epsilon > 0$ is given. Find an n (you only need one) for which the supremum distance between f and f_n is less than $\epsilon/3$. (Such n exist because the f_n converge to f uniformly.) Then, since this f_n is a uniformly continuous function, find a $\delta > 0$ for which $d(f_n(x), f_n(y)) < \epsilon/3$ whenever two points x, y are at most a distance of δ apart. Then I claim that whenever x and y are two points with $d(x, y) < \delta$, we will have $d(f(x), f(y)) < \epsilon$. Simply apply the Triangle Inequality twice:

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

2. You might take, say, $U_n = \left(-\infty, \frac{1}{\sqrt{2}} - \frac{1}{n}\right) \cup \left(\frac{1}{\sqrt{2}} + \frac{1}{n}, \infty\right)$

3. If f and g are both increasing and $x < y$ then $g(x) < g(y)$ so $f(g(x)) < f(g(y))$. You might have been tripped up by the case that both f and g are decreasing: in that case if $x < y$ then $g(x) > g(y)$ but then $f(g(x)) < f(g(y))$, so $f \circ g$ is *increasing*.

4. Apply the Mean Value Theorem from our class to the function $F(x) = \int_a^x f(t)\,dt$. You may recall from Calculus (and we will also prove it) that if f is continuous then F is differentiable and in fact has $F'(x) = f(x)$ for every $x \in [a, b]$. Thus the MVT may be applied and we conclude there is a $c \in (a, b)$ where $f(c) = (F(b) - F(a))/(b - a) = M$. In other words, the so-called "mean value of f on $[a, b]$" is indeed actually a value of f.

Alternatively you may recall that we have proved f attains its maximum and minimum, that is, there are points c and d in $[a, b]$ such that for every $x \in [a, b]$ we have $f(c) \leq f(x) \leq f(d)$. Then the integral $\int_a^b f(x)\,dx$ must lie between the integrals of these two constant functions, which of course equal $(b - a)f(c)$ and $(b - a)f(d)$ respectively. Thus we conclude $f(c) \leq M \leq f(d)$. But that means M is between two values of f; by the Intermediate Value Theorem, M is also a value of f.

5. Use Taylor Series to show that $\sum_{n=1}^{\infty} (-1)^{n-1}/n = \ln(2)$

The Taylor Series of the logarithm function, centered at 1, is the power series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

You learned tests in Calculus that enable you to compute the Radius of Convergence, which is 1; that guarantees the series converges for all $x \in (0, 2)$. That information does not guarantee that the series converges when $x = 2$! (You could establish that fact, too; use the Alternating Series Test.) But even if you know that the series converges for some x, that doesn't tell you what the series converges to! After all, you last week computed the Taylor Series for $e^{-1/x}$, and that series converges for all x but it doesn't converge to the value of the function when $x > 0$!

So you must be more careful. An infinite series like $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ is defined to be the limit of its partial sums S_n (if that limit exists). Those partial sums are the values of the Taylor Polynomials $T_n(x)$ evaluated at $x = 2$. And we have Taylor's Remainder Theorem that tells us how far those partial sums are from $\log(x)$:

$$\log(x) - T_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-1)^{n+1}$$

for some number c between 1 and x. For $f(x) = \log(x)$, that coefficient on the right will be $(-1)^{n-1}/nc^n$

So now if $x > 1$ then $c > 1$ as well, and so the coefficients themselves will tend to zero; if in addition $x \le 2$ then the powers $(x-1)^n$ will tend to 0 or 1. Hence for each $x \in [1, 2]$ we conclude that the differences $\log(x) - T_n(x)$ will tend to zero as $n \to \infty$, meaning that $\log(x) = \lim_n T_n(x)$ for these x, and in particular for $x = 2$ we conclude $\log(2) = \lim T_n(2) = \lim S_n = \sum_{n=1}^{\infty}(-1)^{n-1}/n$ and we are done.

6. (a) When $f(x) = x^{1/3}$ the Newton recurrence is simply $x_{n+1} = -2x_n$, which will obviously not converge unless we start the recurrence right at the solution, i.e. unless we start with $x_0 = 0$.

(b) You may simply quote the Intermediate Value Theorem: if f is continuous on $[a, b]$ and has opposite signs at the endpoints, then f vanishes at some interior point.

(c) If f' is everywhere-positive then f is increasing, meaning whenever $x_1 < x_2$, we have $f(x_1) < f(x_2)$. Thus the two values of f cannot be equal (f must be one-to-one) which means in particular f cannot have two zeros.

(d) Here we simply quote the Taylor Theorem with Remainder: the first of the two equations I presented is the $n = 1$ instance of that theorem, applied to our f; recall that $f(c) = 0$. The second of thet two equations is the $n = 0$ case (the traditional MVT) applied to f'.

(e)-(f) Simply use the two equations of (d) to rewrite the numerator and denominator of the fraction; combine terms as in high school. You can then go back to writing the denominator as $f'(z)$ but the numerator is simply a multiple of $(x-c)^2$.