M365C (Rusin) TEST 1 — Tuesday, Oct 8 2019

Each question is worth 16 points. You will also get 4 points for writing your name and EID:

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1. (a) Prove the following: for all $a, b \in \mathbf{R}$ with $a < b$ there exists a rational multiple of $\sqrt{2}$ between a and b (that is, a number of the form $c\sqrt{2}$ with $c \in \mathbf{Q}$).

(b) Give an example of a subset of $\mathbf{R}$ which is neither open nor closed nor bounded nor compact (and explain why it fails to have each of those properties).

2. Show that the set of all *finite* sets of natural numbers is countable. (Hint: every finite set of natural numbers has a largest element.)

3. In Homework 3 we defined a (total) order on the set of complex numbers in the following way: we said $a + bi < c + di$ iff $(a < c)$ or $(a = c$ and $b < d)$. Find a subset of the complex numbers which has upper bounds in this sense, but has no *least* upper bound.

4. Suppose X is a metric space with metric d, and suppose $x_0 \in X$. Let

$$C = \{x \in X \mid d(x, x_0) \leq 1\}.$$

Show that C is a closed set. (Hint: complements)

Extra Credit: in the special case of $X = \mathbf{Z}$ with the 2-adic metric d, with $x_0 = 0$, show that C is also open!

5. Show that this sequence converges, and determine its limit:

$$2,\ 2+\frac{1}{2},\ 2+\frac{1}{2+\frac{1}{2}},\ 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}},\ \ldots$$

(Hint: to prove convergence, show that the two subsequences consisting of every other term are monotomic.)

6. I asked my calculus students to decide whether the series

$$\sum_{n\geq 1} \frac{(n!)^2}{(2n)!} \left(\frac{2}{5}\right)^n$$

converges. Sadly, some of them don't seem to know the difference between that series and the product

$$\left(\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}\right) \cdot \left(\sum_{n\geq 1} \left(\frac{2}{5}\right)^n\right)$$

But that prompts the following exercise: show that if $\sum a_n$ and $\sum b_n$ are both convergent series of *positive* real numbers, then the series $\sum(a_n b_n)$ converges.