For your viewing pleasure, I will analyze one Partial Fractions problem completely.

Suppose we are asked to find an antiderivative

$$\int \frac{7x-6}{2x^3-x^2+9}\,dx$$

We recognize that the integrand is a rational function, so we breathe a sigh of relief, confident that we can answer this question using Partial Fractions.

First step: the numerator already has a lower degree than the denominator, so no long division is necessary.

Second step: we must factor the denominator into linear and irreducible-quadratic factors. Clearly there are either three linear factors or one linear and one quadratic. Either way there will be real root(s) that will give us linear factors. Let's find those roots.

You know the basic form of the graphs of cubic polynomials: with a positive leading coefficient, this one must either steadily climb, crossing the x-axis once, or else it will hit one hilltop and then one valley along the way. We can discover which it is by looking at its derivative, $6x^2-2x=2x(3x-1)$. This obviously has roots at $x=0$ and $x=1/3$, so yes, there's a hilltop at $(0,9)$ and a valley at $(1/3, 9-1/27)$. But that dip will not give us additional roots: somewhere to the left of $x=0$ is the lone real root.

You could plot a couple more points, say $(x,y)=(-1,6)$ and $(-2,-11)$; that's enough to know (via the Intermediate Value Theorem) that the one root is between -2 and -1.

You could hope that the root is rational, i.e. of the form a/b with a and b integers. Nature gives no *guarantee* that the root must be rational, but you do have my assurance that it will be possible for you to solve the problems I set for you, so let's give it a shot. The Rational Root Theorem says the numerator a would divide 9, so $a \in \{\pm1, \pm3, \pm9\}$; and the denominator b must divide 2. So there are only twelve candidate rational roots. Fortunately for us, we have already decided the root must lie between -2 and -1, and precisely one of these twelve is in that range: $a/b=-3/2$. With fingers crossed we test it and — it works! $x=-3/2$ is a root, which means $x-(-3/2)$ is a factor. (It's OK to clear denominators if you like: $(2x+3)$ will be a factor as well.)

Having identified one factor we use long division to identify the complementary factor $(2x^3-x^2+9)/(2x+3)=(x^2-2x+3)$. We already know this quadratic is irreducible — any additional factors would be linear, which would imply more roots for the cubic, and the graph says otherwise — but you can check the discriminant is negative to confirm this.

With the factorization of the denominator at hand, we decide on the form of the Partial Fractions decomposition. It must be

$$\frac{7x-6}{2x^3-x^2+9}=\frac{A}{2x+3}+\frac{Bx+C}{x^2-2x+3}$$

for some unknown constants A, B, C. (Check: the number of unknowns matches the degree of the denominator.)

Clearing denominators, we see our next task is to find three constants so that these polynomials are identical:

$$7x - 6 = A(x^2 - 2x + 3) + (Bx + C)(2x + 3)$$

Turn this "polynomial" problem into something familiar from high school, namely, turn this into a problem of solving three equations in three unknowns A, B, C. You have several choices of how to get those equations. You could for example compare coefficients of x^2, x, and 1:

$$\begin{aligned} 0 &= A + 2B \\ 7 &= -2A + 3B + 2C \\ -6 &= 3A + 3C \end{aligned}$$

Or, you could plug in various values of x and see what the equations say. For example you could pick any three of these:

$$\begin{array}{rll} -6 & = 3A + 3C & \text{(at } x = 0) \\ 1 & = 2A + 5B + 5C & \text{(at } x = 1) \\ -13 & = 6A - B + C & \text{(at } x = -1) \\ 8 & = 3A + 14B + 7C & \text{(at } x = 2) \\ -33/2 & = (33/4)A & \text{(at } x = -3/2) \end{array}$$

Another option that's handy when when the polynomial has repeated linear factors is to differentiate that polynomial equation and *then* plug in values of x, particularly $x =$ a repeated root. That setting does not apply here but you could follow the steps anyway to get other linear equations: use the product rule to get

$$7 = A(2x - 2) + 2(Bx + C) + B(2x + 3)$$

and then plug in some x values, e.g.

$$\begin{array}{rll} 7 & = -2A + 3B + 2C & \text{(at } x = 0) \\ 7 & = -5A - 3B + 3C & \text{(at } x = -3/2) \end{array}$$

However you do it, create three linear equations and solve them for A, B, C. I find the solution is $A = -2, B = 1, C = 0$. That is, we transformed our original problem into

$$\int \frac{7x - 6}{2x^3 - x^2 + 9}\, dx = \int \left(\frac{-2}{2x + 3} + \frac{x}{x^2 - 2x + 3} \right) dx$$

Now, finally, we can start integrating. Linear denominators are easy. Substitute $u = 2x + 3$ so $du = 2dx$ and the first integral is simply $-\int (1/u)\, du = -\ln(|u|) = -\ln(|2x + 3|) + C$.

The second integral is a bit harder. First complete the square, writing $x^2 - 2x + 3 = (x - 1)^2 + 2$; that suggests a first substitution $u = x - 1$ (so $x = u + 1$ and $du = dx$). Then the integral becomes

$$\int \frac{u + 1}{u^2 + 2}\, du$$

With the linear term in the denominator now gone, this is a good time to split the numerator to get

$$\int \frac{u}{u^2+2}\,du + \int \frac{1}{u^2+2}\,du$$

The first of these is $\frac{1}{2}\ln(u^2+2)$ (substituting $v = u^2+2$ if you like) and the second is $(1/\sqrt{2})\arctan(u/\sqrt{2})$ (substituting $v = u/\sqrt{2}$ if that helps you). Restoring the original variable makes the whole antiderivative into

$$\frac{1}{2}\ln((x-1)^2+2) + \frac{1}{\sqrt{2}}\arctan\left(\frac{(x-1)}{\sqrt{2}}\right) + C$$

Tying everything together, we finally conclude

$$\int \frac{7x-6}{2x^3-x^2+9}\,dx = -\ln(|2x+3|) + \frac{1}{2}\ln(x^2-2x+3) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

Now, wasn't that easy?