How to antidifferentiate

You may think that the book's chapter on antidifferentiation is long and hard but it comes down to just a few ideas.

If you are asked to antidifferentiate a rational function, use Partial Fractions. That's the nickname of a specific recipe that you can follow. There are several steps in this recipe, but none of them is beyond you. Practice them all, and have patience.

If you are asked to antidifferentiate a product of powers of trig functions, use a substitution that will reduce you to the preceding paragraph. Specifically, you should be able to use some trig identities to rewrite your integral in the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

for some whole numbers n and m. You can transform this into a rational-function problem by using one (or more) of these substitutions:

$$\begin{array}{ll} \text{let } u = \sin(x) & \text{if } n \text{ is odd} \\ \text{let } u = \cos(x) & \text{if } m \text{ is odd} \\ \text{let } u = \tan(x) & \text{if } n+m \text{ is even} \end{array}$$

You will have to use the corresponding trig identities. For example in the first case use $du = \cos(x)\,dx$ and then replace $\cos^2(x) = 1 - \sin^2(x)$. The third case is a little more subtle: use $du = \sec^2(x)\,dx$, then use $\cos(x) = 1/\sec(x)$ and $\sin(x) = u/\sec(x)$, and after that replace $\sec^2(x) = 1 + u^2$; in this third case you may find it easier instead to use the double-angle formulas to rewrite even powers of $\sin(x)$ and $\cos(x)$ in terms of lower powers of $\cos(2x)$.

Those are the only two classes of functions for which we expect you to practice a mechanical recipe in order to compute an antiderivative. Look for these: rational functions, and products of trig functions.

If you have to compute an integral which is neither of these, then we expect you instead to use one of two all-purpose tools: Substitution or Integration By Parts. That's all we have for you. Each of them begins with one creative step (the part that says "Let $u =$"). We have tried to give you hints about how to choose the u, as well as hints about which of the two tools to try, but that's all anyone ever gets is some hints. Try something — anything — and if it doesn't work, try something else.

But in each case you *are* responsible for knowing what to do after the creative step. Remember that both tools require you to complete a little diamond of computations (or most of it, anyway) in whichever order seems best to you; then you substitute things in, symbol by symbol, until you get a new antidifferentiation problem. The hope is that the new problem looks better than the original one; if so, go on to use all these tools again; if not, go back to the creative step and try something new.

So let me remind you of the four recipes you should practice.

A. For Partial Fractions, your steps are

1. Divide denominator into numerator if necessary.
2. Factor the denominator.
3. Decide on the form of the Partial Fractions decomposition.
4. Determine the unknown coefficients.
5. Evaluate the antiderivatives of each simple piece. (Note that this last step may require substitutions that lead to trig integrals.)

B. For Trig Integrals, your steps are

1. Eliminate everything but sine and cosine
2. Use a substitution $u = \sin(x)$, $u = \cos(x)$, or $u = \tan(x)$ as appropriate. Use the appropriate Pythagorean identity to get a rational function of u alone to be integrated. (You may prefer to use the Double-Angle formulas instead in that third case, the "even-even" case.)
3. Integrate the resulting rational function. (Note that this may require following the Partial Fractions recipe.)

C. For Substitution, fill in the diamond

$$\begin{array}{ccc} & u = & \\ x = & & du = \\ & dx = & \end{array}$$

1. Pick a substitution "Let $u =$ (some function of x)".
2. Algebraically invert that relationship to get $x =$ (a function of u). (Note that in some cases it's easier to reverse steps 1 and 2.)
3. Differentiate to get a relationship on differentials: either $du = (\ldots)dx$ or $dx = (\ldots)du$; in the former case you also have to solve for dx anyway.
4. Substitute in for dx in your original integral. Then replace all remaining x's with equivalent expressions expressed in terms of u.

This will give you a new antiderivative problem, all expressed in terms of u. Do it.

D. For Integration By Parts, fill in the diamond

$$\begin{array}{ccc} & u = & \\ dv = & & du = \\ & v = & \end{array}$$

1. Pick a substitution "Let $u =$ (some function of x)".
2. Algebraically divide by that relationship to get $dv =$ (a function of x) dx. (Note that in some cases it's easier to reverse steps 1 and 2.)
3. Differentiate to get a relationship on differentials: $du = (\ldots)dx$. You also have to antidifferentiate the result of step 2 to discover a formula for v.
4. Substitute in the results of step 3 into the Integration By Parts formula:

$$\int u\,dv = uv - \int v\,du$$

The left side here is supposed to match your original integration problem; the right side will include a new integral (in terms of x) to compute.

This will give you a new antiderivative problem, all expressed in terms of x. Do it.

I'm obviously leaving out the details here, but that's deliberate: you are supposed to be practicing these four recipes, so I need only to outline the steps for you here. Open your book, or any calculus book under the sun, and find some integration problems to try.

For good measure let me summarize the trig identities you'll use.

A. The definitions $\tan(x) = \sin(x)/\cos(x)$ etc. allow you to throw away everything but sines and cosines, if you wish.

B. The Pythagorean Identities $\sin^2(x) + \cos^2(x) = 1$ and the two that result from dividing this by either $\sin^2(x)$ or $\cos^2(x)$:

$$1 + \cot^2(x) = \csc^2(x) \qquad \text{resp.} \qquad \tan^2(x) + 1 = \sec^2(x)$$

C. Everything else is a consequence of the two Angle Addition Formulas

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \end{aligned}$$

You can divide one by the other and simplify to obtain

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

You can apply to the special case where $b = a$ to get

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

That last equation may be rewritten as either

$$\cos(2a) = 2\cos^2(a) - 1 \qquad \text{or} \qquad \cos(2a) = 1 - 2\sin^2(a)$$

and these equations can be turned around to get useful expressions for $\cos^2(a)$ and $\sin^2(a)$, respectively. In fact letting $a = x/2$ gives what are called the Half-Angle Formulas:

$$\cos(x/2) = \pm\sqrt{\frac{1+\cos(x)}{2}} \qquad \sin(x/2) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

Finally, it may be useful to add or subtract the equations for $\sin(a \pm b)$ or $\cos(a \pm b)$ to get

$$\begin{aligned} \sin(a)\sin(b) &= \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right) \\ \sin(a)\cos(b) &= \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right) \\ \cos(a)\cos(b) &= \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right) \end{aligned}$$