M346 Second Midterm Exam Solutions, October 23, 2003

1. Find all the eigenvalues of the following matrices. You do NOT need to find the corresponding eigenvectors. [Note: the answers are fairly simple, and can be obtained without a lot of calculation, using the various "tricks of the trade".]

a) $\begin{pmatrix} 3 & 1 & 5 & 17 \\ 1 & 3 & 4 & -10 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

This matrix is block triangular, so we just need the eigenvalues of the 2×2 blocks $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, which are 4 and 2, and $2+i$ and $2-i$.

b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

The sum of each row is 6, so one eigenvalue is 6. The determinant is zero (since the first and third rows are identical), so one eigenvalue is zero. The trace is 4, so the third eigenvalue must be -2.

2. The eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ are $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 1$, and corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. (That is, the eigenvalue 1 has multiplicity two, and a basis for the eigenspace E_1 is $\{\mathbf{b}_2, \mathbf{b}_3\}$.)

a) Solve the difference equation $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ (which equals $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, by the way). That is, find $\mathbf{x}(n)$ for every n.

Since $y_1(0) = y_2(0) = y_3(0) = 1$ and $y_k(n) = \lambda_k^n y_k(0)$, we have $\mathbf{x}(n) = 1\lambda_1^n \mathbf{b}_1 + 1\lambda_2^n \mathbf{b}_2 + 1\lambda_3^n \mathbf{b}_3 = (-2)^n \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 = \begin{pmatrix} (-2)^n + 2 \\ (-2)^n - 1 \\ (-2)^n - 1 \end{pmatrix}$.

b) With the situation of part (a), identify the stable, unstable, and neutrally stable modes. What are the limiting ratios $x_1(n)/x_2(n)$ and $x_1(n)/x_3(n)$

when n is large?

Since $|\lambda_1| > 1$ but $|\lambda_2| = |\lambda_3| = 1$, the first mode is unstable (and dominant), while the second and third are neutrally stable. For large n, $\mathbf{x}(n)$ points in the direction of $\mathbf{b}_1$, so the limiting ratios are $1/1 = 1$ and $1/1 = 1$.

c) Now solve the differential equation $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. That is, find $\mathbf{x}(t)$ for all t.

Since $y_1(0) = y_2(0) = y_3(0) = 1$ and $y_k(t) = e^{\lambda_k t} y_k(0)$, we have $\mathbf{x}(t) = e^{-2t}\mathbf{b}_1 + e^t\mathbf{b}_2 + e^t\mathbf{b}_3 == \begin{pmatrix} e^{-2t} + 2e^t \\ e^{-2t} - e^t \\ e^{-2t} - e^t \end{pmatrix}$.

d) With the situation of part (c), identify the stable, unstable, and neutrally stable modes. What are the limiting ratios $x_1(t)/x_2(t)$ and $x_1(t)/x_3(t)$ when t is large?

λ_1 is negative, so the first mode is stable. $\lambda_{2,3}$ are positive, so those modes are unstable. As $t \to \infty$, the e^{-2t} terms go to zero, and we are left with a multiple of $\mathbf{b}_2 + \mathbf{b}_3$, so our ratios go to -2 and -2.

3. Consider the matrix $A = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

Eigenvalues $\lambda_1 = 9$ and $\lambda_2 = -1$, eigenvectors $\mathbf{b}_1 = (1,1)^T$ and $\mathbf{b}_2 = (1,-1)^T$.

b) Write down the general solution to the second-order differential equation $d^2\mathbf{x}/dt^2 = A\mathbf{x}$, with A as above.

The first mode involves hyperbolic trig functions of $\sqrt{\lambda_1}t = 3t$, while the second mode involves ordinary trig functions of $\sqrt{-\lambda_2}t = t$, so our grand total is:

$$\mathbf{x}(t) = [c_1 \cosh(3t) + c_2 \sinh(3t)] \begin{pmatrix} 1 \\ 1 \end{pmatrix} + [c_3 \cos(t) + c_4 \sin(t)] \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

c) Find the solution to this equation when $\mathbf{x}(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\dot{\mathbf{x}}(0) = \begin{pmatrix} 11 \\ 1 \end{pmatrix}$.

The constants $c_1, \ldots, c_4$ are related to the initial conditions by $c_1 = y_1(0)$, $c_2 = \dot{y}_1(0)/3$, $c_3 = y_2(0)$, $c_4 = \dot{y}_2(0)/1$. Since $\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \mathbf{b}_1 + 3\mathbf{b}_2$ and

$\begin{pmatrix} 11 \\ 1 \end{pmatrix} = 6\mathbf{b}_1 + 5\mathbf{b}_2$, we must have $c_1 = 1$, $c_2 = 6/3 = 2$, $c_3 = 3$ and $c_4 = 5/1 = 5$.

4. A 2×2 matrix M has eigenvalues 1 and 8, and corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$.

a) Find $[M]_{\mathcal{B}}$, $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

A matrix, expressed in the basis of its eigenvectors, is diagonal: $[M]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$, while $P = P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ and $P_{\mathcal{B}\mathcal{E}} = P^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$.

b) Find M (expressed in the ordinary basis).

$$M = PDP^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -62 & 42 \\ -105 & 71 \end{pmatrix}.$$

c) A matrix A has the property that $A^3 = M$. Find A. [Hint: what are the eigenvalues and eigenvectors of A?]

The eigenvalues of A are the cube roots of the eigenvalues of M, while the eigenvectors are the same, so $A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -8 & 6 \\ -15 & 11 \end{pmatrix}$.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace E_λ.

True. This is the definition of the geometric multiplicity.

b) If a matrix is diagonalizable, then its eigenvalues are all different.

False. Problem 2 gives a counterexample.

c) Let A by an arbitrary $n \times n$ matrix. The sum of the algebraic multiplicities of the eigenvalues of A must equal n.

True. The sum of the algebraic multiplicities is the degree of the characteristic polynomial, which is n.

d) The eigenvalues of a (square) matrix with real entries are always real.

False. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalue i and $-i$.

e) If $B = PAP^{-1}$, then A and B have the same eigenvalues.

True. They have different eigenvectors but the same eigenvalues.