M346 Second Midterm Exam, November 10, 2005

1. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Compute A^{13421}. (No, you do NOT need a calculator for this!)

c) Compute e^A.

2. a) In $\mathbb{R}^3$ with the standard inner product, apply the Gram-Schmidt process to convert the basis $\left\{(1,4,3)^T, (2,3,4)^T, (10,4,0)^T\right\}$ into an orthogonal basis.

b) Find the coordinates of the vector $\mathbf{v} = (1,-7,9)^T$ in the orthogonal basis you constructed in part (a).

3. Consider the system of difference equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where $A = \begin{pmatrix} -2 & 0 & 0 \\ -5 & 1 & 2 \\ -3 & 1 & 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

a) Diagonalize A.

b) Find $\mathbf{x}(n)$ for all n. (You may express your answer as a linear combination of the eigenvectors of A, but the coefficients should be explicit.)

4. Consider the nonlinear system of differential equations:

$$\frac{dx_1}{dt} = \ln(x_1 x_2^2) \qquad \frac{dx_2}{dt} = \ln(x_1^2 x_2).$$

a) Find the fixed point (there is only one).

b) Find a LINEAR system of ODEs that approximates motion near the fixed point.

c) Find all the stable modes of this linear system. Then find all the unstable modes.

5. a) Find a least-squares solution to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 3 \end{pmatrix}$.

b) Find the equation of the best line through the points $(-1,3)^T$, $(0,1)^T$, $(1,0)^T$, and $(2,-2)^T$.