M346 Second Midterm Exam Solution, November 10, 2005

1. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

The determinant is zero, so 0 must be an eigenvalue. The trace is 1, so the other eigenvalue must be 1. The eigenvectors are $(1,2)^T$ and $(1,1)^T$.

b) Compute A^{13421}. (No, you do NOT need a calculator for this!)

$A = PDP^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $A^{13421} = PD^{13421}P^{-1}$. But $D^{13421} = D$, so this just equals $PDP^{-1} = A$.

c) Compute e^A.

$e^A = Pe^DP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e & 0 \\ 00 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2e-1 & 1-e \\ 2e-2 & 2-e \end{pmatrix}$.

2. a) In $\mathbb{R}^3$ with the standard inner product, apply the Gram-Schmidt process to convert the basis $\left\{(1,4,3)^T, (2,3,4)^T, (10,4,0)^T\right\}$ into an orthogonal basis.

$\mathbf{y}_1 = \mathbf{x}_1 = (1,4,3)^T$, $\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{y}_1 = (1,-1,1)^T$, $\mathbf{y}_3 = \mathbf{x}_3 - \mathbf{y}_1 - 2\mathbf{y}_2 = (7,2,-5)^T$.

b) Find the coordinates of the vector $\mathbf{v} = (1,-7,9)^T$ in the orthogonal basis you constructed in part (a).

$\langle \mathbf{y}_1 | v \rangle = 0$, $\langle \mathbf{y}_2 | \mathbf{v} \rangle / \langle \mathbf{y}_2 | \mathbf{y}_2 \rangle = 17/3$ and $\langle \mathbf{y}_3 | \mathbf{v} \rangle / \langle \mathbf{y}_3 | \mathbf{y}_3 \rangle = -2/3$, so $\mathbf{v} = (17/3)\mathbf{y}_2 - (2/3)\mathbf{y}_3$.

3. Consider the system of difference equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where $A = \begin{pmatrix} -2 & 0 & 0 \\ -5 & 1 & 2 \\ -3 & 1 & 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

a) Diagonalize A.

Eigenvalue -2, 2 and -1, with eigenvectors $(1,1,1)^T$, $(0,2,1)^T$ and $(0,1,-2)^T$. Note that A is block triangular, and that the sum of the entries in each row is -2.

b) Find $\mathbf{x}(n)$ for all n. (You may express your answer as a linear combination of the eigenvectors of A, but the coefficients should be explicit.)

Since $\mathbf{x}(0) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, $\mathbf{x}(n) = (-2)^n\mathbf{b}_1 + 2^n\mathbf{b}_2 + (-1)^n\mathbf{b}_3$.

4. Consider the nonlinear system of differential equations:

$$\frac{dx_1}{dt} = \ln(x_1 x_2^2)$$

$$\frac{dx_2}{dt} = \ln(x_1^2 x_2)$$

a) Find the fixed point (there is only one): The fixed point is $\mathbf{a} = (1,1)^T$.

b) Find a LINEAR system of ODEs that approximates motion near the fixed point.

Setting $\mathbf{y} = \mathbf{x} - (1,1)^T$, we have $d\mathbf{y}/dt \approx A\mathbf{y}$, where $A = \begin{pmatrix} 1/x_1 & 2/x_2 \\ 2/x_1 & 1/x_2 \end{pmatrix}\Big|_{\mathbf{x}=\mathbf{a}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

c) Find all the stable modes of this linear system. Then find all the unstable modes.

The stable mode, with $\lambda = -2$, is $(1,-1)^T$. The unstable mode, with $\lambda = 3$, is $(1,1)^T$.

5. a) Find a least-squares solution to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 3 \end{pmatrix}$.

$A^T A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $A^T\mathbf{b} = \begin{pmatrix} 10 \\ -2 \\ -4 \end{pmatrix}$, so the solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ is $\mathbf{x} = \begin{pmatrix} 5/2 \\ -1/2 \\ -1 \end{pmatrix}$.

b) Find the equation of the best line through the points $(-1,3)^T$, $(0,1)^T$, $(1,0)^T$, and $(2,-2)^T$.

If the equation for the best line is $y = c_0 + c_1 x$, then we are solving the least-squares system $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \end{pmatrix}$. Then $A^T A = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$, and $A^T\mathbf{b} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$, and the solution to $\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$ is $c_0 = 13/10$ and $c_1 = -8/5$. That is, the best line is $y = -1.6x + 1.3$.