A (mod n) Congruence Theorem and its Corollary.

NOTATION NOTE: A string of congruences can be written on one line with the "(mod n)" congruence-type designation at the end of the line.

For example: The statement:

"$77 \equiv 59 \equiv 35 \equiv 5 \pmod 6$" is true and is read as "77 is congruent to 59, which is congruent to 35, which is congruent to 5, all with (mod 6) congruence."

Theorem: For every positive integer n and for every integer a,
$a \equiv (a+n) \pmod n$ and $a \equiv (a-n) \pmod n$.

Proof: Let n be any positive integer and let a be any integer. Then, $a-(a+n) = a-a-n = -n = n\times(-1)$.

$\therefore a-(a+n) = n\times(-1)$.

$\therefore a \equiv (a+n) \pmod n$, by definition of (mod n)-congruence.

Also, $a-(a-n) = a-a+n = n = n\times 1$; $\therefore a-(a-n) = n\times 1$.

$\therefore a \equiv (a-n) \pmod n$, by definition of (mod n)-congruence.

$\therefore$ For every positive integer n and for every integer a,
$a \equiv (a+n) \pmod n$ and $a \equiv (a-n) \pmod n$, by Direct Proof.

QED.

Corollary: For every positive integer n and for every integer a, $a \equiv (a+4n) \pmod{n}$.

Proof: Let n be any positive integer n and let a be any integer.

By multiple applications of the Theorem above,

$a \equiv (a+n) \equiv (a+2n) \equiv (a+3n) \equiv (a+4n) \pmod{n}$.

$\therefore a \equiv (a+4n) \pmod{n}$, by the transitive property of $(\mathrm{mod}\, n)$ congruence.

Therefore, for every positive integer n and for every integer a, $a \equiv (a+4n) \pmod{n}$, by Direct Proof.

QED

Corollary: For every positive integer n and for any two integers a and k,

$$a \equiv (a+kn) \pmod{n}.$$

Proof: The proof is left as an exercise. ∎