Example Proofs Involving the $(n \bmod d)$-function

The following statements are proved in this handout:

(1) To Prove: $(67 \bmod 9) = 4$

(2) To Prove: $(-28 \bmod 5) = 2$

(3) To Prove: For every integer b,
if $(b \bmod 16) = 10$, then $(46 \bmod 16) = 8$.

(4) To Prove: For all integers c and d,
if $(c \bmod 8) = 5$ and $(d \bmod 8) = 7$,
then $(cd \bmod 8) = 3$.

---

(1) To Prove: $(67 \bmod 9) = 4$.

Proof: By the Quotient-Remainder (Q-R) Theorem, there exist unique integers q and r such that $67 = 9q + r$ and $0 \leq r < 9$.

Also, by the definition of $(r \bmod d)$, $(67 \bmod 9) = r$.

Now, $67 = 9 \times 7 + 4$ and $0 \leq 4 < 9$.

So, by the uniqueness of q and r, $q = 7$ and $r = 4$.

$\therefore$ By substitution, $(67 \bmod 9) = 4$.

QED.

(2) To Prove: $(-28 \bmod 5) = 2$

Proof: By the Q-R Theorem, there exist unique integers q and r such that $-28 = 5q + r$ and $0 \leq r < 5$.

By definition of $(n \bmod d)$, $(-28 \bmod 5) = r$.

Now, $-28 = 5 \times (-6) + 2$ and $0 \leq 2 < 5$.

So, by the uniqueness of q and r, $q = -6$ and $r = 2$.

∴ By substitution, $(-28 \bmod 5) = 2$.

QED.

---

(3) To Prove: For every integer b, if $(b \bmod 16) = 10$, then $(4b \bmod 16) = 8$.

Proof: Let b be any integer such that $(b \bmod 16) = 10$.

[N.T.S: $(4b \bmod 16) = 8$].

By the Q-R Theorem, there exist unique integers q_1, r_1 and q_2, r_2 such that $b = 16q_1 + r_1$ and $0 \leq r_1 < 16$ and $4b = 16q_2 + r_2$ and $0 \leq r_2 < 16$.

By definition of $(n \bmod d)$, $(b \bmod 16) = r_1$ and $(4b \bmod 16) = r_2$.

Since $(b \bmod 16) = 10$, $r_1 = 10$ and $b = 16q_1 + 10$, by substitution.

[Proof of (3) continued]

$\therefore 4b = 4(16q_1 + 10)$, by substitution,

$= 64q_1 + 40$

$= 16(4q_1) + 16 \times 2 + 8$

$= 16(4q_1 + 2) + 8$, by Rules of Algebra.

Let $t = 4q_1 + 2$. Then, t is an integer since sums and products of integers are integers.

$\therefore 4b = 16t + 8$, by substitution.

$\therefore 4b = 16t + 8$ and $0 \leq 8 < 16$.

By the uniqueness of q_2 and r_2, $q_2 = t$ and $r_2 = 8$.

As shown above, $(4b \bmod 16) = r_2$.

$\therefore (4b \bmod 16) = 8$, by substitution.

$\therefore$ For every integer b, if $(b \bmod 16) = 10$, Then $(4b \bmod 16) = 8$, by Direct Proof.

QED.

---

(4): To Prove: For all integers c and d, if $(c \bmod 8) = 5$ and $(d \bmod 8) = 7$, then $(cd \bmod 8) = 3$.

Proof: Let c and d be any integers.

Suppose $(c \bmod 8) = 5$ and $(d \bmod 8) = 7$. [Suppose the IF-part!]

[N.T.S.: $(cd \bmod 8) = 3$]

(Continued on the next page)

[Proof of (4) continued]

By the Q-R Theorem there exist unique integers q_1, r_1 and q_2, r_2 and q_3, r_3 such that

$c = 8q_1 + r_1$ and $0 \le r_1 < 8$ and

$d = 8q_2 + r_2$ and $0 \le r_2 < 8$ and

$cd = 8q_3 + r_3$ and $0 \le r_3 < 8$.

By definition of $(n \bmod d)$, $(c \bmod 8) = r_1$, $(d \bmod 8) = r_2$ and $(cd \bmod 8) = r_3$.

Since $(c \bmod 8) = 5$, $r_1 = 5$ and $c = 8q_1 + 5$, by substitution.

Since $(d \bmod 8) = 7$, $r_2 = 7$ and $d = 8q_2 + 7$, by subst.

$\therefore cd = (8q_1 + 5)(8q_2 + 7)$, by substitution,

$= 64q_1q_2 + 40q_2 + 56q_1 + 35$

$= 8(8q_1q_2 + 5q_2 + 7q_1) + 8 \cdot 4 + 3$

$= 8(8q_1q_2 + 5q_2 + 7q_1 + 4) + 3$

$= 8t + 3$, where $t = (8q_1q_2 + 5q_2 + 7q_1 + 4)$,

and t is an integer since sums and products of integers are integers.

$\therefore cd = 8t + 3$ and $0 \le 3 < 8$.

$\therefore$ By the uniqueness of q_3 and r_3, $r_3 = 3$.

As shown above, $(cd \bmod 8) = r_3$. $\therefore$ $(cd \bmod 8) = 3$, by subst.

$\therefore$ For all integers c and d, if $(c \bmod 8) = 5$ and $(d \bmod 8) = 7$, then $(cd \bmod 8) = 3$, by Direct Proof.

QED.