One-to-One Functions & Onto Functions

Official In-the-book Definitions: Let F be a function from a set X to a set Y.

F is *one-to-one* (or *injective*) \Leftrightarrow For every u and v in X,

If
$$F(u) = F(v)$$
, Then $u = v$

Also,

F is *one-to-one* (or *injective*) \Leftrightarrow For every u and v in X,

If
$$u \neq v$$
, Then $F(u) \neq F(v)$.

F is *onto* (or *surjective*) \Leftrightarrow For every element $y \in Y$,

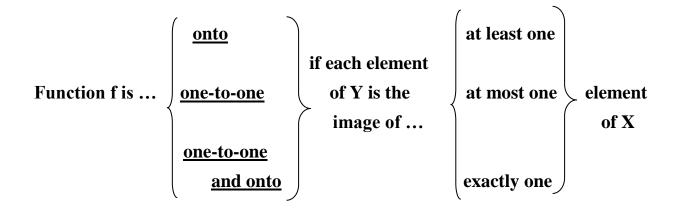
there exists some $x \in X$ such that F(x) = y.

F is a one-to-one correspondence (or a bijection) from **X** to **Y**

 \Leftrightarrow F: X \rightarrow Y is both a one-to-one function and an onto function.

Memorize the above definitions for their use in writing proofs, but a more intuitive definition of these terms is useful and is as follows:

Let $f: X \to Y$ be a function.



If $f: X \to Y$ is one-to-one and onto, then the *inverse function* $f^{-1}: Y \to X$ exists and $f^{-1}(y) = x$ if and only if f(x) = y, for all x in X and all y in Y. **Proof Design I for Proving Function F is One-to-One:**

Function F: $X \rightarrow Y$ is given.

To Prove: Function F is a one-to-one function.

Proof: Suppose that u and v are any two elements of X such that

F(u) = F(v). [We need to show that u = v.]

...
(Using the formula defining F(x) or some other properties
...
of the function F we derive simpler and simpler
...
equations eventually arriving at "u = v".)
∴ u = v.

$$[\therefore \forall u, v \in X, \text{ If } F(u) = F(v), \text{ Then } u = v.]$$

.:. F is one-to-one, by Direct Proof, by Direct Proof. Q E D

Proof Design II for Proving Function F is One-to-One:

Function F: $X \rightarrow Y$ is given.

To Prove: Function F is a one-to-one function.

Proof: Suppose that u and v are any two elements of X such that

 $u \neq v$. [We need to show that $F(u) \neq F(v)$.]

- ... (This is often accomplished using a proof-by-contradiction,
- ... but sometimes it can be shown directly that $F(u) \neq F(v)$.)
- • •

. . .

$$\therefore \mathbf{F}(\mathbf{u}) \neq \mathbf{F}(\mathbf{v}).$$

[$\therefore \forall u, v \in X$, If F(u) = F(v), Then u = v, by contraposition.]

∴ F is one-to-one by Direct Proof. Q E D

Proof Design for Proving that Function F is Onto:

Function F: $X \rightarrow Y$ is given.

To Prove: Function F is an onto function.

Proof: Suppose y is any element in Y.

[We need to show that there is some x in X with F(x) = y.]

(Note: In a workspace, and before the writing of the proof has begun, the equation F(x) = y is manipulated in order to solve for x in terms of y deriving a formula: x = "Formula in terms of y". Use this formula to define the correct pre-image x for the selected y at the start.)

Let x = "Formula in terms of y"

Then, F(x) = (the complicated expression obtained by replacing x by the "Formula in terms of y") = ... (simplifications) ... = y.

[$\therefore \forall y \text{ in } Y$, there exists an element x in X such that F(x) = y.]

: F is onto, by Direct Proof. Q E D

Proof Design to Prove that F is a One-to-One Correspondence (or Bijection):

Function F: $X \rightarrow Y$ is given.

To Prove: F is a One-to-One Correspondence.

Proof:

Part I: [Prove F is one-to-one.] ... ∴ F is one-to-one by Direct Proof . Part II: [Prove F is onto.] … … ∴ F is onto by Direct Proof.

: F is one-to-one and onto.

 \therefore F is a one-to-one correspondence. Q E D