One-to-One Functions & Onto Functions

Official In-the-book Definitions: **Let F be a function from a set X to a set Y.**

F is *one-to-one* (or *injective*) $\Leftrightarrow$ For every u and v in X,

If F(u) = F(v), Then u = v

Also,

F is *one-to-one* (or *injective*) $\Leftrightarrow$ For every u and v in X,

If u $\neq$ v , Then F(u) $\neq$ F(v) .

F is *onto* (or *surjective*) $\Leftrightarrow$ For every element y $\in$ Y,

there exists some x $\in$ X such that F(x) = y .

F is *a one-to-one correspondence* (or a *bijection*) from X to Y

$\Leftrightarrow$ F: X $\rightarrow$ Y is both a one-to-one function and an onto function.

Memorize the above definitions for their use in writing proofs, but a more intuitive definition of these terms is useful and is as follows:

Let f : X $\rightarrow$ Y be a function.

	onto		**at least one**	
Function f is …	**one-to-one**	**if each element of Y is the image of …**	**at most one**	**element of X**
	one-to-one and onto		**exactly one**	

If f : X $\rightarrow$ Y is one-to-one and onto, then the *inverse function* f^{-1} : Y $\rightarrow$ X exists and

f^{-1}(y) = x if and only if f(x) = y , for all x in X and all y in Y.

Proof Design I for Proving Function F is One-to-One:

Function F: $X \to Y$ is given.

To Prove: Function F is a one-to-one function.

Proof: Suppose that u and v are any two elements of X such that

F(u) = F(v) . [We need to show that u = v .]

. . .

. . . (Using the formula defining F(x) or some other properties

. . . of the function F we derive simpler and simpler

. . . equations eventually arriving at "u = v".)

$\therefore$ u = v .

[$\therefore$ $\forall$ u, v $\in$ X , If F(u) = F(v), Then u = v .]

$\therefore$ F is one-to-one, by Direct Proof, by Direct Proof. Q E D

Proof Design II for Proving Function F is One-to-One:

Function F: $X \to Y$ is given.

To Prove: Function F is a one-to-one function.

Proof: Suppose that u and v are any two elements of X such that

u $\neq$ v . [We need to show that F(u) $\neq$ F(v) .]

. . .

. . . (This is often accomplished using a proof-by-contradiction,

. . . but sometimes it can be shown directly that F(u) $\neq$ F(v) .)

. . .

$\therefore$ F(u) $\neq$ F(v).

[$\therefore$ $\forall$ u, v $\in$ X , If F(u) = F(v), Then u = v , by contraposition.]

$\therefore$ F is one-to-one by Direct Proof. Q E D

Proof Design for Proving that Function F is Onto:

Function F: X → Y is given.

To Prove: Function F is an onto function.

Proof: Suppose y is any element in Y.

[We need to show that there is some x in X with F(x) = y .]

(Note: In a workspace, and before the writing of the proof has begun, the equation F(x) = y is manipulated in order to solve for x in terms of y deriving a formula: x = "Formula in terms of y" . Use this formula to define the correct pre-image x for the selected y at the start .)

Let x = "Formula in terms of y"

Then, F(x) = (the complicated expression obtained by replacing x by the "Formula in terms of y") = … (simplifications) … = y.

[∴ ∀ y in Y, there exists an element x in X such that F(x) = y .]

∴ F is onto, by Direct Proof. Q E D

Proof Design to Prove that F is a One-to-One Correspondence (or Bijection):

Function F: X → Y is given.

To Prove: F is a One-to-One Correspondence.

Proof:

Part I: [Prove F is one-to-one.] … … ∴ F is one-to-one by Direct Proof .

Part II: [Prove F is onto.] … … ∴ F is onto by Direct Proof.

∴ F is one-to-one and onto .

∴ F is a one-to-one correspondence. Q E D