Proving that (0, 1) is an Uncountable Set

Theorem (NIB) 14:

Any infinite sequence of real numbers, all of which are between 0 and 1, will fail to include at least one real number between 0 and 1. That is:

For any infinite sequence $b_1 , b_2 , b_3 , \ldots$ such that, $0 < b_{\mathbf{i}} < 1$ for every $\mathbf{i} \in \mathbb{Z}^+$,

there exists some real number z , $0 < z < 1$, such that $b_{\mathbf{i}} \neq z$ for all $\mathbf{i} \in \mathbb{Z}^+$.

Proof:

Suppose $b_1 , b_2 , b_3 , \ldots$ is any infinite sequence of real numbers such that, for all $\mathbf{i} \in \mathbb{Z}^+$, $0 < b_{\mathbf{i}} < 1$.

Line up the $b_{\mathbf{i}}$'s in a column with their decimal expansions, and ensure that in the decimal expansions, those expansions that end with 99999… are replaced by equivalent expansions that end with 00000…

(each $\mathbf{d}_{\mathbf{i}\,\mathbf{j}}$ is the $\mathbf{j}^{th}$ digit in the expansion of $b_{\mathbf{i}}$ as found in the $\mathbf{i}^{th}$ row):

$$
\begin{array}{l}
b_1 = 0\ .\ d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15}\ d_{16}\ d_{17}\ \ldots \\
b_2 = 0\ .\ d_{21}\ d_{22}\ d_{23}\ d_{24}\ d_{25}\ d_{26}\ d_{27}\ \ldots \\
b_3 = 0\ .\ d_{31}\ d_{32}\ d_{33}\ d_{34}\ d_{35}\ d_{36}\ d_{37}\ \ldots \\
b_4 = 0\ .\ d_{41}\ d_{42}\ d_{43}\ d_{44}\ d_{45}\ d_{46}\ d_{47}\ \ldots \\
b_5 = 0\ .\ d_{51}\ d_{52}\ d_{53}\ d_{54}\ d_{55}\ d_{56}\ d_{57}\ \ldots \\
\vdots \qquad\qquad\qquad \vdots
\end{array}
$$

A real number $\mathbf{z}$ will be constructed so that $0 < z < 1$ and such that $\mathbf{z}$ does not appear in the sequence $b_1 , b_2 , b_3 , \ldots$.

To define this number **z** , we will focus on the $\mathbf{i}^{th}$ digit, $d_{\mathbf{ii}}$, of the $\mathbf{i}^{th}$ term $b_{\mathbf{i}}$ for each $\mathbf{i} \in \mathbb{Z}^{+}$:

$$
\begin{array}{lllllllll}
b_1 = 0\,. & \boxed{d_{11}} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} & \ldots \\
b_2 = 0\,. & d_{21} & \boxed{d_{22}} & d_{23} & d_{24} & d_{25} & d_{26} & d_{27} & \ldots \\
b_3 = 0\,. & d_{31} & d_{32} & \boxed{d_{33}} & d_{34} & d_{35} & d_{36} & d_{37} & \ldots \\
b_4 = 0\,. & d_{41} & d_{42} & d_{43} & \boxed{d_{44}} & d_{45} & d_{46} & d_{47} & \ldots \\
b_5 = 0\,. & d_{51} & d_{52} & d_{53} & d_{54} & \boxed{d_{55}} & d_{56} & d_{57} & \ldots \\
\quad\vdots & & & & \vdots & & & &
\end{array}
$$

Then, we define this real number **z** as follows:

$$\mathbf{z} = 0.\, a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \;\ldots\;, \text{ where, for each } \mathbf{i} \in \mathbb{Z}^{+} ,$$

$$a_i = \begin{cases} 5 & \text{if } d_{ii} \neq 5 \\ 7 & \text{if } d_{ii} = 5 \end{cases}$$

By this process, the number $z = 0.\, a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \;\ldots$ is uniquely defined.

The digits in the decimal expansion of z consist of 5's and 7's and the choice of each digit as 5 or 7 depends on the digits in the decimal expansions of the particular numbers in the sequence $b_1 , b_2 , b_3 , \ldots$.

For example, suppose the sequence $b_1 , b_2 , b_3 , . . .$ begins as follows:

b_1	=	0	.	**3**	8	2	6	7	5	2	. . .
b_2	=	0	.	4	**6**	1	9	5	8	5	. . .
b_3	=	0	.	9	3	**5**	8	6	1	2	. . .
b_4	=	0	.	2	5	0	**9**	4	3	7	. . .
b_5	=	0	.	5	8	7	2	**5**	9	0	. . .
	.						.				
	.						.				
	.						.				

Thus, the decimal expansion for z will begin as shown:

$$\mathbf{z = 0.55757...}$$

Now, it can be seen that $\mathbf{z \neq b_1}$
because their expansions differ in the 1^{st} digit: $d_{11} = 3$, whereas $a_1 = 5$.

It can be seen that $\mathbf{z \neq b_2}$
because their expansions differ in the 2^{nd} digit: $d_{22} = 6$, whereas $a_2 = 5$.

It can be seen that $\mathbf{z \neq b_3}$
because their expansions differ in the 3^{rd} digit: $d_{33} = 5$, whereas $a_3 = 7$.

In the same way, **for all** $\mathbf{i} \in \mathbb{Z}^+$**,** $\mathbf{z \neq b_i}$ because their expansions differ in the i^{th} digit,

$a_i \neq d_{ii}$.

If the sequence $b_1 , b_2 , b_3 , . . .$ is different, the number z will be different,
but it will still be true that every digit in the decimal expansion of z is 5 or 7
(which guarantees that $0 < z < 1$)
and it will still be true that z differs from every number in the sequence.

Thus, in the general case, for the arbitrarily chosen sequence $b_1 , b_2 , b_3 , . . . ,$
there exists a real number z such that $0 < z < 1$ and $b_i \neq z$, for all $i \in \mathbb{Z}^+$.

Q E D

Theorem (NIB) 15: The interval (0 , 1) = $\{ x \in \mathbb{R} \mid 0 < x < 1 \}$ is uncountable.

Proof: (proof-by-contradiction)

The interval (0 , 1) is either finite or infinite.

Certainly (0 , 1) is not finite because it contains the countably infinite set $\{ 1/2 , 1/3 , 1/4 , 1/5 , \ldots \}$ as a subset.

$\therefore$ The interval (0 , 1) is either a countably infinite set or an uncountable set.

Suppose that (0 , 1) is not uncountable, by way of contradiction.

Then, (0 , 1) is countably infinite.

$\therefore$ There exists a one-to-one correspondence $f : \mathbb{Z}^+ \to (0 , 1)$.

[Note: f is one-to-one and onto; in particular, f is onto.]

Define the infinite sequence $b_1 , b_2 , b_3 , \ldots$ as follows:

For all $i \in \mathbb{Z}^+$, $b_i = f(i)$. Then, for all $i \in \mathbb{Z}^+$, $0 < b_i < 1$.

By Theorem (NIB) 14, there is some real number z , $0 < z < 1$, such that z does not appear in the sequence $b_1 , b_2 , b_3 , \ldots$;

that is, for all $i \in \mathbb{Z}^+$, $b_i \neq z$. But that means that for all $i \in \mathbb{Z}^+$, $f(i) \neq z$.

Thus, f is not onto, which contradicts the assumption that f is a one-to-one correspondence. $\therefore$ The interval (0 , 1) is uncountable. Q E D

Theorem (NIB) 16: For Y equal to the set of all points on the Unit Circle minus the North Pole (0,1) , the set Y is uncountable.

Proof: There are one-to-one correspondences between the Unit Circle minus the North Pole and the Interval (0 , 1) of real numbers. One such one-to-one correspondence is $f : (0 , 1) \to Y$, defined as follows: For all $t \in (0 , 1)$,

$$f(t) = \left(\cos \left(2\pi t + \frac{\pi}{2} \right) , \sin \left(2\pi t + \frac{\pi}{2} \right) \right) .$$

Therefore, interval (0 , 1) and Y have the same cardinality.

Since (0 , 1) is uncountable by Theorem (NIB) 15, Y is uncountable. Q E D