In-the-Book Definitions (II)

Set U is the Universal Set. Any sets discussed are subsets of the Universal Set U.

Set A is *non-empty* ($A \neq \varnothing$) $\Leftrightarrow$ There exists an element $x \in U$ such that $x \in A$.

The Union of set A and set B is the set $A \cup B$ where:

$$A \cup B = \{ x \in U \mid x \in A \text{ OR } x \in B \} .$$

The *procedural* definition of $A \cup B$ which is useful for writing proofs involving $A \cup B$ is:

$$x \in A \cup B \Leftrightarrow x \in A \text{ OR } x \in B .$$

Whenever a set X is defined in the form

$$X = \{ x \in U \mid \text{Predicate P is true about } x \},$$

the *procedural* definition of X is: $x \in X \Leftrightarrow$ Predicate P is true about x .

The Intersection ($\cap$) and Difference ($-$) of sets A and B, and the Complement (A^c) of set A, are defined as follows:

$$A \cap B = \{ x \in U \mid x \in A \text{ AND } x \in B \} ; \qquad A^c = \{ x \in U \mid x \notin A \};$$

$$A - B = \{ x \in U \mid x \in A \text{ AND } x \notin B \}$$

Set A *is a subset of* Set B ($A \subseteq B$) $\Leftrightarrow$ For all elements $x \in U$, if $x \in A$, then $x \in B$.

A collection $\{ A_1, A_2, \dots A_n \}$ of non-empty subsets of A is a Partition of set A

$\Leftrightarrow$ 1) $A = A_1 \cup A_2 \cup \dots \cup A_n$ and 2) when $i \neq j$, $A_i \cap A_j = \varnothing$.

Given a set A, the Power Set of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.
When A is finite with n elements, then $\mathcal{P}(A)$ has 2^n elements.

Set Equality: Set A = Set B $\Leftrightarrow$ $A \subseteq B$ and $B \subseteq A$.

A (Binary) Relation R from set A to Set B is any subset of the Cartesian Product $A \times B$.

Given such a relation R, for all $a \in A$ and $b \in B$, $a R b \Leftrightarrow (a, b) \in R$.

The Inverse Relation R^{-1} is the subset of $B \times A$ such that
for all $b \in B$ and $a \in A$, $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$; thus, $b R^{-1} a \Leftrightarrow a R b$.

A Relation R on A is a relation R from A to A, that is, from A to B with B = A .

Let R be a binary relation on set A:

1) R is <u>reflexive</u> $\Leftrightarrow$ $\forall x \in A$, $x R x$.
2) R is <u>symmetric</u> $\Leftrightarrow$ $\forall x, y \in A$, IF $x R y$, THEN $y R x$.
3) R is <u>transitive</u> $\Leftrightarrow$ $\forall x, y, z \in A$, IF $x R y$ and $y R z$, THEN $x R z$.

An <u>Equivalence Relation</u> is a relation which is reflexive, symmetric and transitive.

Given the Equivalence Relation R on set A and given element *a* in set A, the <u>Equivalence Class of *a*</u> (or just the *Class of a*) is denoted $[\, a \,]$ and is defined as $[\, a \,] = \{ x \in A \mid x R a \}$.

Any element z in an equivalence class for equivalence relation R is called an <u>Equivalence Class Representative</u> of that class, and in this case the equivalence class containing the representative z will be the set $[\, z \,]$.

A <u>Complete Set of Equivalence Class Representatives</u> is a set containing exactly one representative from each of the equivalence classes of R.

A <u>Function f from Set X to Set Y</u>, denoted $f : X \to Y$, is a binary relation from X to Y such that both:

1) For every element x in X, there is some element y in Y with $(x, y) \in f$, that is, $x f y$ for at least one element y in Y, and

2) For every element x in X and all elements y and z in Y, if $x f y$ and $x f z$, then $y = z$, that is, f relates each x in X to only one element y in Y.

Here, the set X is called the <u>domain of f</u>; the set Y is the <u>co-domain of f</u>.

For a given $x \in X$, there is a unique $y \in Y$ with $x f y$, and this element y is called "the image of x under f", or also, "the value of the function f at x", or also " f of x ", and we write " $y = f(x)$ " to be read as "y equals f of x."

The <u>Range of f</u> is the set of all images of f in the set Y ; that is,

<u>Range of f</u> $= \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}$.

The (Range of f) $\subseteq$ Co-domain Y, but it can happen that the (Range of f) $\neq$ Y.

Given an element y in Y, the <u>inverse image of y in X under f</u> is the set:

inverse image of y $= \{ x \in X \mid f(x) = y \}$.

For all $y \in Y$, $y \in$ (Range of f) $\Leftrightarrow$ inverse image of y under f is not $\emptyset$.

For all subsets $A \subseteq X$ and all subsets $C \subseteq Y$,

the <u>image of A under f (or f of A)</u> is $f(A) = \{ y \in Y \mid y = f(x)$ for some $x \in A \}$, and

the <u>inverse image of C (or f inverse of C)</u> is $f^{-1}(C) = \{ x \in X \mid f(x) \in C \}$.

<u>Equality of Functions:</u> Suppose $f : X \to Y$ and $g : X \to Y$ are functions from X to Y.
Then <u>f equals g</u> ($f = g$) $\Leftrightarrow$

1) f and g have the same domains and co-domains, and

2) $\forall x \in X$, $f(x) = g(x)$.

Given set X , define the <u>identity function</u> i_X by the rule: $\forall z \in X$, $i_X(z) = z$.

Let f be a function from a set X to a set Y.
f is <u>*one-to-one* (or *injective*)</u> $\Leftrightarrow$ For every u and v in X,
If $f(u) = f(v)$, Then $u = v$
$\Leftrightarrow$ For every u and v in X,
If $u \neq v$, Then $f(u) \neq f(v)$.

f is <u>*onto* (or *surjective*)</u> $\Leftrightarrow$ For every element $y \in Y$,
there exists some $x \in X$ such that $f(x) = y$.

f is a <u>*one-to-one correspondence* (or a *bijection*)</u> from X to Y
$\Leftrightarrow$ $f : X \to Y$ is both a one-to-one function and an onto function.

Given that $f : X \to Y$ is a one-to-one correspondence from X to Y,
then the inverse relation f^{-1} is also a function , $f^{-1} : Y \to X$, and is such that

$$\forall y \in Y, \ f^{-1}(y) = x \Leftrightarrow f(x) = y .$$

Composition of Functions:

Given functions f and g, where $f: X \to V$ and $g: Y \to Z$, where X, Y, V, and Z are any sets, such that the (range of f) $\subseteq$ (domain of g) = Y ,

the Composition of f and g (written $g \circ f$) is the function defined by the rule:

$$\forall\ x \in X,\quad g \circ f(x) = g(f(x)) \text{ in } Z.$$

(We can refer to the composition of f and g as " g circle f , " and, given a particular value of x, we can say, " g circle f of x equals g - of - f - of - x . ")

Note:
If $f: X \to Y$ and $g: Y \to Z$ are two functions such that f and g are both one-to-one, then $g \circ f: X \to Z$ is also one-to-one. (Theorem 7.3.3)

If $f: X \to Y$ and $g: Y \to Z$ are two functions such that f and g are both onto, then $g \circ f: X \to Z$ is also onto. (Theorem 7.3.4)

If $f: X \to Y$ and $g: Y \to Z$ are two functions such that f and g are both one-to-one and onto (i.e., f and g are one-to-one correspondences) , then $g \circ f: X \to Z$ is also one-to-one and onto (i.e., $g \circ f$ is also a one-to-one correspondence).

If $f: X \to Y$ is a one-to-one correspondence, then the inverse function, $f^{-1}: Y \to X$, Is also a one-to-one correspondence, and (using i_X and i_Y for the identity functions),

$$f^{-1} \circ f = i_X \quad \text{and} \quad f \circ f^{-1} = i_Y$$

that is: $\forall\ x \in X,\ f^{-1} \circ f(x) = f^{-1}(f(x)) = x = i_X(x)$, and

$$\forall\ y \in Y,\ f \circ f^{-1}(y) = f(f^{-1}(y)) = y = i_Y(y) .$$

Having the SAME CARDINALITY:
Set A has the same cardinality as Set B if there is a one-to-one correspondence from A to B. A and B have the same cardinality if A has the same cardinality as B and B has the same cardinality as A.

For a non-empty Set X :
Set X is infinite $\Leftrightarrow$ $\forall n \in \mathbb{Z}^+$, X and { 1, 2, . . . , n } do not have the same cardinality.
Set X is countably infinite $\Leftrightarrow$ X and $\mathbb{Z}^+$ have the same cardinality.
Set X is countable $\Leftrightarrow$ X is finite OR X is countably infinite.
A set which is not countable is said to be uncountable.