Predicate Calculus

Definition: A Predicate (Open Sentence) is a sentence that contains one or more variables, and the sentence becomes a logical statement when values are substituted for the variables.

Examples: P(x): "x^2 is positive."
Q(x,y): "x is the daughter of y"

P(x) is true when x represents 3.
P(x) is false when x represents 0.

The Domain of a variable is the set of values that can replace that variable, i.e., the values that the variable can represent.

The Truth Set of predicate P(x) is the set of all values in the domain of x for which P(x) is true when x is replaced by those values. The Truth Set = $\{ x \in \text{Domain } D \mid P(x) \text{ is true} \}$

A Quantifier is a phrase added to a predicate which makes an assertion concerning the number of values in the domain of the variable that make the predicate true.

The Universal Quantifier ($\forall$) asserts that every value in domain makes the attached predicate true.

Phrases: "For all $x \in D$, P(x)." ; "For every $x \in D$, P(x)." ; "For any $x \in D$, P(x)." In Symbols: $\forall x \in D, P(x)$.

These statements are called Universal Statements.

The Existential Quantifier (∃) asserts that at least one value in domain makes the attached predicate true.

Phrases: "There exists a value $x \in D$, such that $P(x)$." ;
"For at least one $x \in D$, $P(x)$." ;
"For some $x \in D$, $P(x)$." ; "There is a value for x ..."
In Symbols: $\exists x \in D$ such that $P(x)$.

These statements are called Existential Statements.

Truth Values of Quantified Statements

The universal statement with form "$\forall x \in D$, $P(x)$" is defined to be True if, and only if, $P(x)$ is true for every value for x in the domain D. It is False when there is at lease one value for x in D such that $P(x)$ is false for that value, and any particular value of x which makes $P(x)$ false is called a counterexample of the universal statement.

The existential statement with form "$\exists x \in D$ such that $P(x)$" is defined to be True if, and only if, $P(x)$ is true for at least one value for x in the domain D. It is False when $P(x)$ is false for every value for x in D.

The declaration and definition of a variable made within a universal statement is applied locally and only within that universal statement.

The declaration and definition of a variable made within an existential statement is applied globally, except when used as the predicate of a universal statement.

Negations of Universal and Existential Statements

The negation of the universal statement "$\forall x \in D, P(x)$" is
"$\exists x \in D$ such that $\sim P(x)$."

Thus, $\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $\sim P(x)$.

So, the negation of a universal statement is a particular existential statement asserting the existence of a counterexample making the predicate false.

The negation of existential statement "$\exists x \in D$ such that $P(x)$." is "$\forall x \in D, \sim P(x)$."

Thus, $\sim(\exists x \in D$ such that $P(x)) \equiv \forall x \in D, \sim P(x)$.

So, the negation of an existential statement is a particular universal statement asserting the lack of any example value making the predicate true.