Defining Variables in a Proof

(The following discussion concerns some of the conventions used in writing and reading mathematical proofs.)

A **variable** is a symbol used to represent a particular mathematical or logical entity with a particular (but sometimes unspecified) value.

Examples of variables: **x, y, z, f, P, m, Temp, Trigfunc, p, q, r, s**

Examples of Mathematical Entities:
number, function, circle, set, point, integer, real number, rational number, positive real number, negative integer, non-negative integer, natural number, irrational number, ordered pair

Examples of Logical Entities:
statement, predicate, premise, conclusion, existential quantifier, universal quantifier, conditional statement, bi-conditional statement, equivalence, conjunction, disjunction

A **Variable Definition** is a statement stating that a certain symbol will be used to represent a particular mathematical or logical entity, stated in such a manner that the reader is able to determine unambiguously whether any given value is one which the variable could represent or is one which the variable could not represent. The **domain of a variable** is the set of all values which that variable could represent. The definition of a variable must define unambiguously the domain of that variable as a set.

Examples of Direct Definitions of Variables:

"The symbol **x** will represent any real number."
"The symbols **m, n,** and **r** will represent any integers."

More usual ways to phrase these two statements are:
"Let **x** be a real number."
"Suppose that **m, n,** and **r** are integers."

In this class, every direct definition of a variable must use the "Let …" option above, stated **in the order**: "Let <*variable*> be <*value description*>" or "Let <*variable*> = <*value formula*>."
Stating the definition in the order, "Let <*value formula*> = "<*variable*>," is not allowed.

Examples of Adequately Defined Variables (including First-use Definition and Inherited Declarations.)

(It is assumed that the symbols used as variables here have not appeared earlier in the proof.)

"The symbol x will represent the integer 1,736,423."
(Entity-type is **integer** and particular value is 1,736,423.)

"Let x = 1,736,423." (Entity-type is **integer** and particular value is 1,736,423.)

“Let x be a particular positive integer whose value is not specified.”
(Entity type is positive integer and its particular value is one of the following: 1, 2, 3, …)

(Note: The particularity of the value of x is adequately specified because it is unambiguous that, for instance, x cannot represent 0 or –3 , but x can represent 5 or 3,688,932.)

Shorthand for this statement: “Let x be **any** positive integer.”
An equivalent shorthand for this statement: “Let x be **a** positive integer.”

(So: “Let x be … ” means “The symbol x represents …” and “Let x be **a** …” means “Let x be **any** …”)

“Let **x** be a positive integer and let **y** = **x** – 100 .”

(The definition of **x** here is an example of a **First-use Definition**. The implied definition of **y** as an integer is an example of an **Inherited Definition**.)

Here the symbol **y** is defined to be the integer whose value is 100 less than the value which **x** represents. Since the value of **y** is calculated in terms of **x** and **x** has been defined already as an integer and the calculation defining **y** results in another integer, then **y** inherits the definition as an integer also. Relying on inherited definitions is dangerous and should be avoided because they are easy to state incorrectly or to interpret incorrectly. For instance, without thinking about it carefully, one might assume (incorrectly) that the variable **y** has been defined as a positive integer because **x** has been defined as a positive integer. However, when the value of **x** is 100 or less, the value of **y** is negative or is 0 and therefore not positive. It would be better to state:
“Let **x** be a positive integer and let **y** be the integer such that **y** = **x** – 100 .”
This way, the definition of **y** as an integer is made explicitly and cannot be incorrectly interpreted.

Example of Inadequately Defined Variables:

(It is assumed that the symbols used as variables here have not appeared earlier in the proof.)

“Let **y** = **x / 2**.”

The variable **x** has not been defined earlier and is not adequately defined here, so it is not clear whether **x** is an integer, a real number, or a complex number. Consequently, it is unclear whether **y** inherits a definition as another rational number, another real number, or another complex number.

A variable has been adequately defined if, at the point in a proof when the symbol is used, it can be unambiguously determined what the set of values that the symbol might represent is (that is, what the domain of the variable is).

Global Variable and Local Variable Definitions:

A variable is **Globally Defined** if the declaration of its entity-type and the setting of its value is preserved throughout an argument (unless its value is reset by a global redefinition).

A variable is **Locally Defined** if the declaration of its entity-type and the setting of its value is meaningful for only a limited section of an argument, after which the symbol loses the previous definition and must be redefined before it can meaningfully be used again.

The examples of variable definitions presented so far have all been Global Variable Definitions.

Several occasions when variables are locally defined will be encountered during the semester. For now, we look at one of the most common use of Locally Defined Variables:

When a variable is defined within an IF–THEN statement, its definition applies only within that particular IF–THEN statement alone. The use of that variable beyond that IF–THEN statement, without adequate redefinition of that variable, is incorrect and lacks unambiguous meaning.

Example:

In the following argument excerpt, the definition of the variable **x** as an even positive integer applies only within the IF–THEN statement. After this statement, the symbol **x** loses its definition as a positive integer, or as an integer, or indeed as a number of any kind.

"If **x** is an even positive integer, then (**x + 2**) is also an even positive integer.
Let **y** be such that **y = (x + 2) / 2**.
Therefore, **y** is an integer because the division of an even integer by 2 results in the calculation of an integer."

The above argument is meaningless although it might not seem so to you now. Since variable **x** is defined as representing an even positive integer in the IF-THEN statement, the variable **x** loses the definition as representing an even positive integer when it is used in the second sentence. Since **x** loses its definition as an even positive integer, it is not adequately defined as representing any particular mathematical entity in the second sentence, and so **y** is not adequately defined because there is no entity definition of **x** for **y** to inherit.

This point about x losing its definition when it is used outside of its defining IF-THEN statement is subtle, but it is important for you to understand.

It might be more easily understood after making the following considerations:

In the following, no declaration of variable **x** of any kind, globally or locally, is made:

"Let **y** be such that **y = (x + 2) / 2**.
Therefore, **y** is an integer."

In that case, it is pretty clear that variable **x** is used without adequate definition of what it represents. In fact, the conclusion that **y** is an integer is incorrect when **x** is 1, which we might imagine is the case, lacking an adequate definition of **x**.

In the following, an interesting fact about numbers is added at the start, but still there is no definition of the variable **x**:

"If the number 2 is added to any even positive integer, then the result is another even positive integer.
Let **y** be such that **y = (x + 2) / 2**.
Therefore, **y** is an integer."

In that case also, the variable **x** is used without adequately defining what it represents (What if **x** = 1?).

Now, **an example of defining the variable x within an IF–THEN statement**:

"If **x** is an even positive integer, then (**x + 2**) is also an even positive integer."

is just another way of saying:

"If the number 2 is added to any even positive integer, then the result is another even positive integer."

Thus, the above argument can be stated equivalently (and incorrectly) as follows:

"If **x** is an even positive integer, then (**x + 2**) is also an even positive integer.
Let **y** be such that **y = (x + 2) / 2**.
Therefore, **y** is an integer."

Here, the variable **x** is only defined within the IF-THEN statement containing its definition as an even positive integer. Outside of the IF-THEN statement in which it was defined, the variable **x** becomes an undefined variable, unavailable for use until it is redefined. Consequently, the variable **y** is also an undefined variable.

The argument excerpt can be corrected by adding a statement redefining the variable **x** as a global variable:

"If **x** is an even positive integer, then (**x + 2**) is also an even positive integer.
Let **x** be any even positive integer.
Let **y** be such that **y = (x + 2) / 2**.
Therefore, **y** is an integer because the division of an even integer by 2 results in the calculation of an integer."

>> IT IS AN ERROR TO USE AN INADEQUATELY DEFINED VARIABLE IN A PROOF.<<<

Example Exercises:

In the following proof segments, identify which variables are inadequately defined and explain in each case why the variable is inadequately defined, and then make sufficient changes or additions to the argument so that all the variables are adequately defined.
If every variable used is adequately defined, then so state.

Proof Segment 1)

"Let **x** be a real number greater than 2.
If **y** is a real number greater than 5, then **(x + y)** is greater than 7.
Therefore, **y** is greater than 3 since 5 is greater than 3."

Solution for Proof Segment 1)

The variable **y**, when it is used in the third sentence, is not adequately defined. When **y** is defined in the second sentence, its definition within an IF–THEN statement is local to that sentence and so **y** is defined only within that statement. When the symbol **y** is used in the third statement, without **y** being redefined, it is being used without definition..

Correction:

"Let **x** be a real number greater than 2.
If **y** is a real number greater than 5, then **(x + y)** is greater than 7.
Let **y** be a real number greater than 5.
Therefore, **y** is greater than 3 since 5 is greater than 3."

Proof Segment 2)

"If **x** is an integer, then **(x + 3)** is an integer.
Let **y** be a positive integer.
Let **z** = **w + y**.
Therefore, **z** is greater than **w**, since **y** is a positive integer.."

Solution for Proof Segment 2)

Both variables **z** and **w** are inadequately defined. In the third sentence, variable **w** is used without previously being defined. The variable **z** is defined in terms of variable **w**, which has not been defined as representing any particular entity, so **z** inherits no particular definition as representing a particular entity.

Correction:

"If **x** is an integer, then **(x + 3)** is an integer.
Let **y** be a positive integer.
Let **w** be any integer.
Let **z** = **w + y**.
Therefore, **z** is greater than **w**."

Proof Segment 3)

"Let **x** be a real number greater than 2.
Let **y** be a positive integer.
Let **z** = **x + y**.
Therefore, **z** is greater than **x**."

Solution for Proof Segment 3)

Every variable used is adequately defined.
There are no places in which an inadequately defined variable is used.

End of Example Exercises

Exercises:

In the following proof excerpts, identify which variables are inadequately defined and explain in each case why each inadequately defined variable is inadequately defined, and then make sufficient changes or additions to the argument so that all the variables are adequately defined.

Note: Your corrected argument may not be a valid argument in that the conclusion may be false, but **your corrected argument must not use inadequately defined variables.**

Also, **if every variable used in the argument excerpt is adequately defined, then state**: **"All variables used are adequately defined."**

1) "Let **x** = 4.
Let **y** = 6.
Let **z = x + y.**
Therefore, **z** = 10."

2) "Let **x** be an integer less than 5.
Let **z** = **y** + 3.
Let **w = x + z.**
Let **q** = **x** + 3.
Therefore, **x** > 20."

3) "Let **y** be a positive integer.
If **x** is 4, then **(x + y)** > **(y + 3).**
If **z** is 6, then **(z + y)** > **(y + 5).**
Let **w** = **x + z.**
Therefore, **w** = 10**."**

4) "Let **a = 1; let b = 3**, and let **c** be a negative integer.
If **d** is an integer less than **c**, then **d** is negative.
Let **y** be any real number greater than 1,000.
Let **z** = **x + y.**
Let **k** = **a + c.**
Let **m** = **b + c.**
Let **n** = **d + c.**
Therefore, **k + m** = **2c + 4.**"

5) "Let **y** be any real number greater than 1,000.
Let **x** be any integer less than –1,000.
Let **z** be greater than 500 but less than 1,000.
Therefore, **(y + x – z)** is a negative number."

6) "Let **t** > 5.
Suppose **y** is an integer such that **y** > **t.**
Let **x** = **t** + 10.
Therefore, **x** > **y** ."