HW 10B, PART I Solutions

M325K
SPRING 2024

SEC 8.3

#4

$A = \{a, b, c, d\}$

$R = \{(a,a), (b,b), (b,d), (c,c), (d,b), (d,d)\}$

$[a] = \{a\}$, $[b] = \{b, d\}$

$[c] = \{c\}$

#6 $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

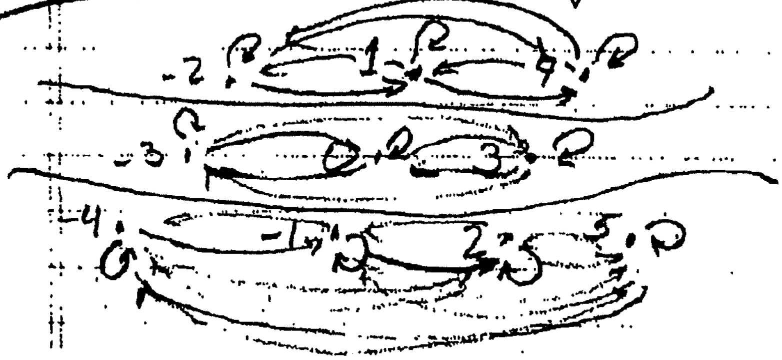
$\forall x,y \in A,\ x R y \Leftrightarrow 3 \mid (x-y)$

$[0] = \{-3, 0, 3\}$ $\quad -3R0,\ 0R0,\ 3R0$

$[1] = \{-2, 1, 4\}$ $\quad -2R1,\ 1R1,\ 4R1$

$[2] = \{-4, -1, 2, 5\}$ $\quad -4R2,\ -1R2,\ 2R2,\ 5R2$

Optional The Directed Graph of R:



Sec 8.3 #10

$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

$\forall x, y \in A,\ xRy \Leftrightarrow 3 \mid (x^2 - y^2)$

$3 \mid [(\pm 5)^2 - 1^2] = 24$; $3 \mid [(\pm 4)^2 - 1^2] = 15$

$3 \mid [(\pm 2)^2 - 1^2] = 3$; $3 \mid [(\pm 1)^2 - 1^2] = 0$

$\therefore\ \pm 5 R 1,\ \pm 4 R 1,\ \pm 2 R 1,\ \pm 1 R 1$

$3 \mid [(\pm 3)^2 - 3^2] = 0$, so $\pm 3 R 3$

$3 \mid [0^2 - 0^2] = 0$, so $0 R 0$.

$\therefore [0] = \{0, 3, -3\}$

$[1] = \{-5, -4, -2, -1, 1, 2, 4, 5\}$

Sec. 8.3

#13. (a) $17 - 2 = 15 = 5 \times 3$
$\therefore 5 \mid (17-2)$ $\quad \therefore 17 \equiv_{(\text{mod } 5)} 2$

(a) $\therefore 17 \equiv 2 \pmod 5$ is TRUE.

(b) $4 - (-5) = 20$. $\frac{20}{7}$ is not in $\mathbb{Z}$.
$\therefore 7 \nmid (4-(-5))$
$\therefore 4 \not\equiv_{(\text{mod } 7)} -5$

(b) $\therefore 4 \equiv -5 \pmod 7$ is False.

(c) $-2-(-8) = 6 = 3 \times 2$
$\therefore 3 \mid (-2-(-8))$. $\quad \therefore -2 \equiv_{(\text{mod } 3)} -8$

(c) $\therefore -2 \equiv -8 \pmod 3$ is TRUE.

(d) $-6-(22) = -28 = 2 \times (-14)$
$\therefore 2 \mid (-6-22)$ $\quad \therefore -6 \equiv 22_{(\text{mod } 2)}$.

(d) $\therefore -6 \equiv 2 \pmod 2$ is TRUE.

Sec 8.3

#14 In solving both parts of #14, we make repeated uses of Lemma 8.3.2, which states:

Lemma 8.3.2: Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If aRb, then $[a]=[b]$.

(a) Write R for the "congruence modulo 3" relation. The only R-EQUIVALENCE CLASSES ARE $[0]$, $[1]$, $[2]$.

$7-1=6=3\times 2$. $\therefore 3 \mid 7-1$. $\therefore 7 \equiv 1 \pmod 3$

$\therefore [7]=[1]$ by Lemma 8.3.2.

Also, $4R1$ since $3 \mid (4-1)=3$. $\therefore [4]=[1]$

and $19R1$ since $3 \mid (19-1)=18$. $\therefore [19]=[1]$

$-4-2=-6=3(-2)$. $\therefore 3 \mid -4-2$. $\therefore -4 \equiv 2 \pmod 3$

$\therefore [-4]=[2]$ by Lemma 8.3.2.

Similarly, $17R2$ and $[17]=[2]$.

$-6-0=-6=3(-2)$. $\therefore 3 \mid (-6-0)$. $\therefore -6 \equiv 0 \pmod 3$

$\therefore [-6]=[0]$ by Lemma 8.3.2.

Similarly $27R0$ and $[27]=[0]$.

$\therefore [-6]=[27]$ (Both equal $[0]$)

$\therefore [7]=[4]=[19]$ (all 3 equal $[1]$)

$\therefore [-4]=[17]$ (Both equal $[2]$.)

Sec 8.3, #14 (cont.)

(b) Write R for the "Congruence modulo 7" relation.

For "congruence modulo 7", the only equivalence classes are $[0], [1], [2], [3], [4], [5], [6]$.

$35 - 0 = 35 = 7 \times 5$. $\therefore 7 \mid 35-0$, $\therefore 35 \equiv 0 \pmod 7$

$\therefore [35] = [0]$.

Similarly, $-7 R 0$ and $[-7] = [0]$.

$\therefore [35] = [-7] = [0]$.

$17 - 3 = 14 = 7 \times 2$; $\therefore 7 \mid 17-3$; $\therefore 17 \equiv 3 \pmod 7$

$\therefore [17] = [3]$

$12 - 5 = 7 = 7 \times 1$; $\therefore 7 \mid 12-5$; $\therefore 12 \equiv 5 \pmod 7$

$\therefore [12] = [5]$.

Similarly, $-2 - 5 = -7 = 7(-1)$.

$\therefore 7 \mid (-2-5)$, $\therefore -2 \equiv 5 \pmod 7$

$\therefore [-2] = [5]$. $\therefore [12] = [-2]$.

$[35] = [-7] = [0]$ (all three equal $[0]$.)

$[17] = [3]$ (Both equal $[3]$)

$[12] = [-2]$ (Both equal $[5]$.)

Sec 8.3, #19 (NOT ASSIGNED)

Define the relation F on $\mathbb{Z}$ by requiring that, for all $m, n \in \mathbb{Z}$,

$$mFn \Leftrightarrow 4 \mid (m-n).$$

To Prove: F is an equivalence relation

Proof: [F is reflexive]

Let x be any integer. Then, $x - x = 0 = 4 \times 0$.
$\therefore 4 \mid (x-x)$. $\therefore xFx$, by def'n of relation F.
$\therefore F$ is reflexive, by direct proof.

[F is symmetric]

Let x and y be any integers. Suppose xFy. [NTS: yFx]
Then, $4 \mid (x-y)$, by def'n of relation F. $\therefore x - y = 4k$ for some integer k.
$\therefore (y-x) = (-1)(x-y) = (-1)(4k) = 4(-k)$.
$\therefore 4 \mid (y-x)$. $\therefore yFx$, by def'n of relation F.
$\therefore F$ is symmetric, by direct proof.

[F is transitive] Let x, y and z be integers such that xFy and yFz.
[NTS: xFz]. $\therefore$ By def'n of F, $4 \mid (x-y)$ and $4 \mid (y-z)$.
$\therefore$ There exist integers k and l such that $(x-y) = 4k$ and $(y-z) = 4l$. Now, $(x-z) = (x-y) + (y-z)$.
$\therefore x - z = 4k + 4l$, by substitution. $\therefore x - z = 4(k+l)$.
$\therefore 4 \mid (x-z)$. $\therefore xFz$, by def'n of relation F.
$\therefore F$ is transitive by direct proof.

$\therefore F$ is reflexive, symmetric and transitive. $\therefore F$ is an equivalence relation. QED

Sec 8.3, #19 (Continued)

The EQUIVALENCE CLASSES ARE:

$[0] = \{n \in \mathbb{Z} \mid n = 4k \text{ for some integer } k\}$

$[1] = \{n \in \mathbb{Z} \mid n = 4k+1 \text{ for some integer } k\}$

$[2] = \{n \in \mathbb{Z} \mid n = 4k+2 \text{ for some integer } k\}$

$[3] = \{n \in \mathbb{Z} \mid n = 4k+3 \text{ for some integer } k\}$

Another way to write these:

$[0] = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}$

$[1] = \{\ldots, -7, -3, 1, 5, 9, 13, \ldots\}$

$[2] = \{\ldots, -6, -2, 2, 6, 10, 14, \ldots\}$

$[3] = \{\ldots, -5, -1, 3, 7, 11, 15, \ldots\}$

SECTION 8.3, #21:

A is the "absolute value" relation defined on $\mathbb{R}$ as follows:
For all $x, y \in \mathbb{R}$, $\quad x A y \iff |x| = |y|$.

(1) To Prove: A is an equivalence relation.
Proof:
[We first prove that A is reflexive.]
Let $x \in \mathbb{R}$ be given.
$|x| = |x|$.
$\therefore$ $x A x$ by definition of the relation A.
$\therefore$ A is reflexive, by Direct Proof.

[We next prove that A is Symmetric.]

Let $x, y \in \mathbb{R}$ be given.
Suppose $x A y$.
Then, $|x| = |y|$ by def'n of relation A.
$\therefore$ $|y| = |x|$ by rules of algebra.
$\therefore$ $y A x$, by definition of relation A.
$\therefore$ A is symmetric, by Direct Proof.

[We next prove that A is transitive.

Let $x, y, z \in \mathbb{R}$ be given.
Suppose $x A y$ and $y A z$.
Then, $|x| = |y|$ and $|y| = |z|$ by definition of relation A

Section 8.3, #21 (cont.)

$\therefore |x| = |z|$ by the transitivity of "=".

$\therefore x A z$, by the definition of relation A.

$\therefore$ For all $x, y, z \in \mathbb{R}$,

if $x A y$ and $y A z$, then $x A z$, by Direct Proof.

$\therefore$ A is transitive, by def'n of "transitive relation".

Since A is reflexive, symmetric and transitive,
A is an equivalence relation.

QED.

#24. For any real number x, define the set I_x as follows:

$$I_x = \{ \text{all } y \in \mathbb{R} \mid \text{there exists an integer } k \text{ such that } y = x + k \}$$

The Distinct Equivalence Classes are
all the sets I_x such that $x \in \mathbb{R}$ and $0 \le x < 1$.

An example of one such equivalence class is $I_{\left(\frac{\sqrt{2}}{2}\right)}$, where

$$I_{\left(\frac{\sqrt{2}}{2}\right)} = \left\{ \ldots \frac{\sqrt{2}}{2} - 2, \frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} + 1, \frac{\sqrt{2}}{2} + 2, \ldots \right\}$$

Section 8.3, #39

The proof is incorrect.

The error is that, for a given element x in A, to be able to use the symmetric and transitive properties to conclude that xRx, there must first exist an element y in A such that xRy.

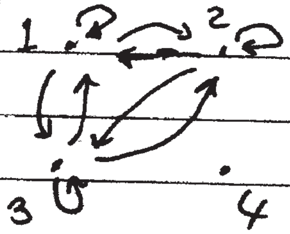
If there does not exist such an element y, then the element x must not be related to any element of A, not even itself.

So, a counterexample relation showing that the theorem is incorrect is one which has an element $x \in A$ such that $x \not R y$, for all $y \in A$.

Counterexample: Let $A = \{1, 2, 3, 4\}$

Let $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

It's Direct Graph is:



$4 \in A$ and $4 \not R y$, for all $y \in A$

The Relation R is not reflexive since $4 \not R 4$,

but R is symmetric and transitive.