M325K SPRING 2024

HW #12 Section 7.2 SOLUTIONS

SEC 7.2:

#1: The two statements are logically equivalent because the second is the contrapositive of the first.

#2 (a): for "one-to-one": AT MOST (b) for "onto": AT LEAST

#8a H is not one-to-one because b and c are in the domain X such that $b \neq c$, but $H(b) = H(c)$.

H is not onto, because there exist elements, such as z, in the co-domain Y such that $H(x) \neq z$ for all $x \in X$.

#8b K is one-to-one because, for all $u, v \in X$, if $u \neq v$, then $K(u) \neq K(v)$.
Here, $K(a) \neq K(b)$, $K(a) \neq K(c)$; $K(b) \neq K(c)$

K is not onto, there there exist elements, such as z, in the co-domain Y such that $K(x) \neq z$ for all $x \in X$.

Sec. 7.2,

#10(a).

Define $f: \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2n$, for all $n \in \mathbb{Z}$.

(i) Is f one-to-one? Yes!

To Prove: f is one-to-one.

Proof:

Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ be given.

Suppose $f(m) = f(n)$.

[NTS: $m = n$]

$\therefore$ By def'n of f, $2m = 2n$.

$\therefore$ $m = n$, by rules of algebra.

$\therefore$ For all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, if $f(m) = f(n)$, then $m = n$, by Direct Proof.

$\therefore$ f is one-to-one, by def'n of one-to-one.

QED.

(ii) Is f onto? NO.

For example, $y = 3$ is an integer and is a counterexample.

Suppose $f(x) = 3$, for some integer x.

Then, $2x = 3$, by def'n of f. $\therefore x = 1.5$, which contradicts the fact that x is an integer.

$\therefore$ For all integers x, $f(x) \neq 3$, by Proof by Contradiction. $\therefore$ f is not onto. QED.

Sec 7.2

#17 Let $f(x) = \frac{3x-1}{x}$ for all real numbers x with $x \neq 0$.

To Prove: Function f is one-to-one.

Proof: Let u and v be any two real numbers such that $u \neq 0$ and $v \neq 0$.

Suppose that $f(u) = f(v)$. [NTS: $u = v$].

By definition of f, $f(u) = \frac{3u-1}{u}$ and

$f(v) = \frac{3v-1}{v}$. $\therefore \frac{3u-1}{u} = \frac{3v-1}{v}$.

Since $u \neq 0$ and $v \neq 0$, $uv \neq 0$.

Multiplying both sides by uv, $(3u-1)v = u(3v-1)$

$\therefore 3uv - v = 3uv - u$. [Next, subtract $3uv$.]

$\therefore -v = -u$. [Next, multiply by -1]

$\therefore v = u$

$\therefore u = v$.

$\therefore$ For all real numbers u, v with $u \neq 0$ and $v \neq 0$, if $f(u) = f(v)$, then $u = v$, by Direct Proof.

$\therefore f$ is one-to-one. QED.

Note: With function f defined as above, the domain of f is $X = \mathbb{R} - \{0\}$ and the codomain Y seems to be $\mathbb{R}$.

However, this function is poorly defined because the codomain of f is not specified precisely.

SECTION 7.2

#25 Define $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and $G: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows:

For all $(n,m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$,

$F((n,m)) = 3^n 5^m$ and

$G((n,m)) = 3^n 6^m$.

#25 (a): Is F one-to-one? Yes.

Proof that F is one-to-one.

Suppose u_1 and u_2 are elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that

$F(u_1) = F(u_2)$. [NTS: $u_1 = u_2$]

By def'n of $\mathbb{Z}^+ \times \mathbb{Z}^+$, there are positive integers, n_1, m_1, n_2, m_2 in $\mathbb{Z}^+$ such that

$u_1 = (n_1, m_1)$ and $u_2 = (n_2, m_2)$.

By definition of F and since $F(u_1) = F(u_2)$,

$F((n_1, m_1)) = F((n_2, m_2))$, and so

$3^{n_1} \cdot 5^{m_1} = 3^{n_2} \cdot 5^{m_2}$, with $n_1 > 0$, $n_2 > 0$, $m_1 > 0$ and $m_2 > 0$.

Let $N = 3^{n_1} \cdot 5^{m_1} = 3^{n_2} \cdot 5^{m_2}$.

Then N is an integer such that $N > 1$.

Since 3 and 5 are prime numbers, $N = 3^{n_1} 5^{m_1}$ and $3^{n_2} \cdot 5^{m_2}$ are both prime factorizations of N.

By the UNIQUE PRIME FACTORIZATION THEOREM, Theorem 4.3.5,
$n_1 = n_2$ and $m_1 = m_2$.

$\therefore$ As ordered pairs, $(n_1, m_1) = (n_2, m_2)$.

$\therefore$ $u_1 = u_2$ since $u_1 = (n_1, m_1)$ and $u_2 = (n_2, m_2)$.

$\therefore$ F is a one-to-one function, by direct proof.
QED.

#25 (b): Is G one-to-one? YES.

Proof that G is one-to-one.
Suppose (n_1, m_1) and (n_2, m_2) are elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that
$G((n_1, m_1)) = G((n_2, m_2))$.

Then $3^{n_1} 6^{m_1} = 3^{n_2} 6^{m_2}$ by def'n of G.

(Sec 7.2, # 25 Continued)

Let $N = 3^{n_1} 6^{m_1} = 3^{n_2} 6^{m_2}$.

Then, since $6 = 2 \times 3$,

$N = 3^{n_1} 6^{m_1} = 3^{n_1} \times 2^{m_1} \times 3^{m_1} = 2^{m_1} \times 3^{(n_1+m_1)}$

and

$N = 3^{n_2} 6^{m_2} = 3^{n_2} \times 2^{m_2} \times 3^{m_2} = 2^{m_2} \times 3^{(n_2+m_2)}$

Since 3 and 2 are prime numbers,

$N = 2^{m_1} 3^{(n_1+m_1)}$ and $N = 2^{m_2} 3^{(n_2+m_2)}$

are prime factorizations of N, and $N > 1$,

By The Unique Prime Factorization Theorem, Theorem 4.3.5,

$m_1 = m_2$ and $(n_1+m_1) = (n_2+m_2)$

$\therefore$ By substitution $n_1 + m_1 = n_2 + m_1$

[We just substituted m_1 for m_2 in the equation "$(n_1+m_1) = (n_2+m_2)$"]

$\therefore$ [Subtracting m_1 from both sides] $n_1 = n_2$.

Since $n_1 = n_2$ and $m_1 = m_2$, $(n_1, m_1) = (n_2, m_2)$.

$\therefore$ G is one-to-one, by direct proof.

QED

SECTION 7.2

#48 The function f is a one-to-one correspondence because it is one-to-one and onto.

Let $X = \mathbb{R} - \{0\}$ Let $Y = \mathbb{R} - \{3\}$.

Define function $f: X \to Y$ as follows:

For all $x \in X$, $f(x) = \frac{3x-1}{x}$.

Proof that f is one-to-one:
This was accomplished in the solution for problem #17 of Section 7.2.

Proof that f is onto:

WORKSPACE: Given y, if $f(x) = y$, then

$$\frac{3x-1}{x} = y \Rightarrow 3x-1 = xy \Rightarrow 3x - xy = 1$$

$$\Rightarrow x(3-y) = 1 \Rightarrow x = \frac{1}{3-y}$$

Continuing the proof that f is onto:

Let y be any element of $Y = \mathbb{R} - \{3\}$

(Sec 2.2, #48, continued)

Then $y \in \mathbb{R}$ and $y \neq 3$.

Let $x = \frac{1}{3-y}$. Then, x is well-defined since $y \neq 3$, and clearly $x \neq 0$. $\therefore x \in X = \mathbb{R} - \{0\}$

$$f(x) = \frac{3x-1}{x}$$

$$\therefore f(x) = \frac{3\left(\frac{1}{3-y}\right) - 1}{\left(\frac{1}{3-y}\right)} = \frac{\frac{3}{3-y} - \frac{(3-y)}{(3-y)}}{\left(\frac{1}{3-y}\right)}$$

$$\therefore f(x) = \frac{(3-3+y)/(3-y)}{(1)/(3-y)} = \frac{y/(3-y)}{1/(3-1)}$$

$$\therefore f(x) = \left(\frac{y}{\cancel{3-y}}\right)\left(\frac{\cancel{(3-y)}}{1}\right) = \frac{y}{1} = y.$$

$\therefore f(x) = y$. $\therefore$ For all $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$.

$\therefore f$ is onto. QED.

By Direct proof.

$\boxed{\text{Define } f^{-1}: Y \to X \text{ as follows: For all } y \in Y,\ f^{-1}(y) = \frac{1}{3-y}.}$