M325K SPRING 2024

HW #13B, PART II SOLUTION

Define function $g: \mathbb{R} \to \mathbb{R}$ as follows:
For all $x \in \mathbb{R}$, $g(x) = x^5 + 30$.

To Prove: g is a one-to-one correspondence.

[Workspace: Discovering $h(y)$
so that $h \circ g = i_{\mathbb{R}}$ and $g \circ h = i_{\mathbb{R}}$.
Given $x \in \mathbb{R}$, we need $y \in \mathbb{R}$ and $h(y) = x$ where
$y = g(x)$. $y = g(x) \Leftrightarrow y = x^5 + 30 \Leftrightarrow$
$\Leftrightarrow x^5 = y - 30 \Leftrightarrow x = \sqrt[5]{y-30} = h(y)$]

Proof: Define function $h: \mathbb{R} \to \mathbb{R}$ as follows:
For all $y \in \mathbb{R}$, $h(y) = \sqrt[5]{y - 30}$.

[$h \circ g = i_{\mathbb{R}}$] Note: $h \circ g : \mathbb{R} \to \mathbb{R}$ and $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, so the Domains of the two functions are the same and the co-domains of the two functions are the same.

Let $x \in \mathbb{R}$ be given

$$h \circ g(x) = h(g(x)) = h(x^5 + 30) = \sqrt[5]{(x^5+30) - 30}$$
$$= \sqrt[5]{x^5} = x = i_{\mathbb{R}}(x).$$

$\therefore h \circ g(x) = i_{\mathbb{R}}(x)$.

$\therefore$ For all $x \in \mathbb{R}$, $h \circ g(x) = i_{\mathbb{R}}(x)$, by Direct Proof.

$\therefore h \circ g = i_{\mathbb{R}}$

[$g \circ h = i_{\mathbb{R}}$] Note: $g \circ h : \mathbb{R} \to \mathbb{R}$ and $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, so the Domains of the two functions are the same and the Co-domains of the two functions are the same.

Let $y \in \mathbb{R}$ be given.

$$g \circ h(y) = g(h(y)) = g(\sqrt[5]{y-30})$$
$$= (\sqrt[5]{y-30})^5 + 30 = (y-30)+30$$
$$= y = i_{\mathbb{R}}(y).$$

$\therefore$ $g \circ h(y) = i_{\mathbb{R}}(y)$.

$\therefore$ For all $y \in \mathbb{R}$, $g \circ h(y) = i_{\mathbb{R}}(y)$, by Direct Proof.

$\therefore$ $g \circ h = i_{\mathbb{R}}$.

Thus, $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions such that $g \circ h = i_{\mathbb{R}}$ and $h \circ g = i_{\mathbb{R}}$.

$\therefore$ By Theorem (NIB) 10, g is a one-to-one correspondence.

QED.