HW #13, SECTION 7.3 SOLUTIONS

M325K
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#5. Define $f: \mathbb{R} \to \mathbb{R}$ by the rule:

For all $x \in \mathbb{R}$, $f(x) = -x$.

FIND the formula for $f \circ f(x)$.

Let $x \in \mathbb{R}$ be given

$$f \circ f(x) = f(f(x)) = f(-x) = -(-x) = x.$$

$\therefore$ For all $x \in \mathbb{R}$, $f \circ f(x) = x$.

#8. Define $L: \mathbb{Z} \to \mathbb{Z}$ and $M: \mathbb{Z} \to \mathbb{Z}$ by the rules:

For all integers a, $L(a) = a^2$ and $M(a) = (a \bmod 5)$.

(a) $L \circ M(12) = L(M(12)) = L((12 \bmod 5)) = L(2)$
$= 2^2 = 4$. $\quad \underline{L \circ M(12) = 4}$

$M \circ L(12) = M(L(12)) = M(12^2) = M(144)$
$= (144 \bmod 5) = 4 \quad \underline{M \circ L(12) = 4}$

$L \circ M(9) = L(M(9)) = L((9 \bmod 5)) = L(4) = 4^2 = 16$
$\underline{L \circ M(9) = 16}$

$M \circ L(9) = M(L(9)) = M(9^2) = M(81) = (81 \bmod 5) = 1$
$\underline{M \circ L(9) = 1}$

(b) It is true that functions L and M have the same domains and the same co-domains,
$L \circ M(9) = 16$ and $M \circ L(9) = 1$.

$\therefore$ $L \circ M \neq M \circ L$, as functions.

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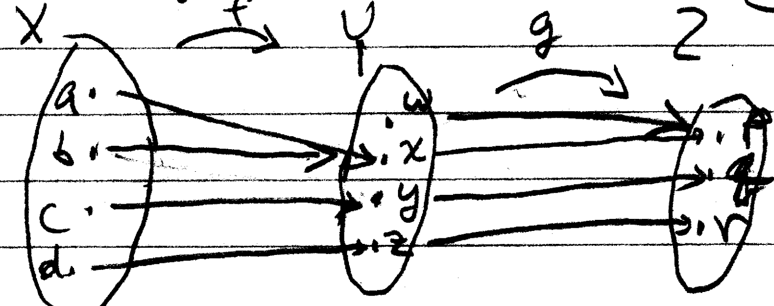
#17 If $f: X \to Y$ and $g: Y \to Z$ are functions such that $g \circ f$ is onto, it is possible that f is NOT onto, so it is not true that both f and g must be onto.

As a counter-example:

Let $X = \{a, b, c, d\}$, $Y = \{w, x, y, z\}$
$Z = \{p, q, r\}$

We define $f: X \to Y$ and $g: Y \to Z$ so that their arrow diagrams are the following:



Define: $f(a) = x$, $f(b) = x$, $f(c) = y$, $f(d) = z$
$g(w) = p$, $g(x) = p$, $g(y) = q$, $g(z) = r$

$g \circ f(a) = g(f(a)) = g(x) = p$
$g \circ f(c) = g(f(c)) = g(y) = q$
$g \circ f(d) = g(f(d)) = g(z) = r$.

$\therefore$ $g \circ f$ is an onto function.

However, $f(x) \neq w$ for all $x \in X$, so f is not onto.

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#18 Let X, Y and Z be sets and let $f: X \to Y$ and $g: Y \to Z$ be functions.

To Prove: If $g \circ f$ is one-to-one, then f is one-to-one.

Proof: Suppose that $g \circ f$ is one-to-one.
$g \circ f: X \to Z$

Let u, v be elements of X.
Suppose that $f(u) = f(v)$. [NTS: $u = v$.]

Since $f(u) = f(v)$ in the set Y,
$g(f(u)) = g(f(v))$ by substitution.

$\therefore g \circ f(u) = g \circ f(v)$ by substitution
$\therefore$ Since $g \circ f$ is one-to-one, $u = v$.
$\therefore$ For all $u, v \in X$, if $f(u) = f(v)$, then $u = v$, by direct proof.

$\therefore f$ is one-to-one by definition of "one-to-one".
QED.