HW #1.3

Section 7.4 Solutions

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(48) **#4**

Let $\mathcal{O}$ = the set of all odd integers
$2\mathbb{Z}$ = the set of all even integers

THEN:

$\mathcal{O} = \{ n \in \mathbb{Z} \mid n = 2k+1 \text{ for some } k \in \mathbb{Z} \}$

$2\mathbb{Z} = \{ n \in \mathbb{Z} \mid n = 2k \text{ for some } k \in \mathbb{Z} \}$.

To Prove: $\mathcal{O}$ and $2\mathbb{Z}$ have the same cardinality.

Proof:

Define function $f: 2\mathbb{Z} \to \mathcal{O}$ as follows:

For all $x \in 2\mathbb{Z}$, $f(x) = x+1$

Note: For all $x \in 2\mathbb{Z}$, $x = 2k$ for some $k \in \mathbb{Z}$ and so, $f(x) = x+1 = 2k+1 \in \mathcal{O}$.

Thus, f is well-defined.

Define function $g: \mathcal{O} \to 2\mathbb{Z}$ as follows:

For all $y \in \mathcal{O}$, $g(y) = y-1$.

Note: For all $y \in \mathcal{O}$, $y = 2k+1$ for some $k \in \mathbb{Z}$ and so $g(y) = y-1 = (2k+1)-1 = 2k \in 2\mathbb{Z}$.

Thus, g is well-defined.

Let $x \in 2\mathbb{Z}$ be given

$g \circ f(x) = g(f(x)) = [f(x)] - 1 = (x+1)-1 = x = i_{2\mathbb{Z}}(x)$,

$\therefore$ For all $x \in 2\mathbb{Z}$, $g \circ f(x) = i_X(x)$, by direct proof.

$\therefore$ $g \circ f = i_X$

(Sec 7.4 #4 continued)

Let $y \in \mathbb{O}$ be given

$f \circ g(y) = f(g(y)) = [g(y)] + 1 = (y-1)+1 = y = i_{\mathbb{O}}(y)$

$\therefore$ For all $y \in \mathbb{O}$, $f \circ g(y) = i_{\mathbb{O}}(y)$, by Direct Proof. $\therefore f \circ g = i_{\mathbb{O}}$.

Since $g \circ f = i_{2\mathbb{Z}}$ and $f \circ g = i_{\mathbb{O}}$, f is a one-to-one correspondence from $2\mathbb{Z}$ to $\mathbb{O}$ by Theorem (NIB) 10,

$\therefore$ $\mathbb{O}$ and $2\mathbb{Z}$ have the same cardinality.

QED.

Sec 7.4

<u>#9</u> <u>To Prove</u>: $\mathbb{Z}^{\text{NON-NEGATIVE}}$, the set of all non-negative integers, is countable.

<u>Proof</u>: $\mathbb{Z}^{\text{NON-NEG}} = \{ n \in \mathbb{Z} \mid n \geq 0 \}$

[We first prove that $\mathbb{Z}^{\text{NON-NEG}}$ is countably infinite]

Define function $f: \mathbb{Z}^{\text{NON-NEG}} \longrightarrow \mathbb{Z}^{+}$ as follows:

For all $x \in \mathbb{Z}^{\text{NON-NEG}}$, $f(x) = x+1$.

Define function $g: \mathbb{Z}^{+} \rightarrow \mathbb{Z}^{\text{NON-NEG}}$ as follows:

For all $y \in \mathbb{Z}^{+}$, $g(y) = y-1$.

Clearly, f and g are well-defined.

For all $x \in \mathbb{Z}^{\text{NON-NEG}}$

$g \circ f(x) = g(f(x)) = (f(x)) - 1 = (x+1) - 1 = x = i_{\mathbb{Z}^{\text{NON-NEG}}}(x)$

$\therefore g \circ f = i_{\mathbb{Z}^{\text{NON-NEG}}}$.

For all $y \in \mathbb{Z}^{+}$,

$f \circ g(y) = f(g(y)) = (g(y)) + 1 = (y-1)+1 = y = i_{\mathbb{Z}^{+}}(y)$.

$\therefore f \circ g = i_{\mathbb{Z}^{+}}$.

Since $g \circ f = i_{\mathbb{Z}^{\text{NON-NEG}}}$ and $g \circ f = i_{\mathbb{Z}^{+}}$,

f is a one-to-one correspondence from $\mathbb{Z}^{\text{NON-NEG}}$ onto $\mathbb{Z}^{+}$.

$\therefore \mathbb{Z}^{\text{NON-NEG}}$ is countably infinite.

$\therefore \mathbb{Z}^{\text{NON-NEG}}$ is countable.

QED

Sec 7.4, #9, Alternate Solution.

To prove: $\mathbb{Z}^{\text{non-NEGATIVE}}$ the set of all non-negative integers, is countable.

Proff: Define $f: \mathbb{Z}^{\text{non-neg}} \to \mathbb{Z}^+$ as follows:

For all $x \in \mathbb{Z}^{\text{non-neg}}$, $f(x) = x + 1$.

Since $\mathbb{Z}^{\text{non-neg}} = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 1\}$,

it is clear that f is well-defined.

[Showing that f is one-to-one.].

Let $u, v \in \mathbb{Z}^{\text{non-neg}}$ be given.

Suppose $f(u) = f(v)$. By definition of f, $f(u) = u+1$

and $f(v) = v+1$.

$\therefore$ $u+1 = v+1$ by substitution.

$\therefore$ $u = v$, by Rules of Algebra.

$\therefore$ For all $u, v \in \mathbb{Z}^{\text{non-neg}}$, if $f(u) = f(v)$,

then $u = v$, by Direct Proof.

$\therefore$ f is one-to-one, by definition of "one-to-one function."

[Showing that f is onto.]

Let $y_0 \in \mathbb{Z}^+$ be given. Then $y_0 \geq 1$, by def'n of $\mathbb{Z}^+$.

$\therefore$ $y_0 - 1 \geq 0$, so $y_0 - 1 \in \mathbb{Z}^{\text{non-neg}}$.

Let $x_0 = y_0 - 1$. $\therefore$ $f(x_0) = x_0 + 1 = (y_0 - 1) + 1 = y_0$.

$\therefore$ For all $y \in \mathbb{Z}^+$, there exists an $x \in \mathbb{Z}^{\text{non-neg}}$ such that

$f(x) = y$, by Direct proof.

$\therefore$ f is onto, by def'n of "onto function".

$\therefore$ Function $f: \mathbb{Z}^{\text{non-neg}} \to \mathbb{Z}^+$ is a one-to-one correspondence.

$\therefore$ $\mathbb{Z}^{\text{non-neg}}$ and $\mathbb{Z}^+$ have the same cardinality. $\therefore$ $\mathbb{Z}^{\text{non-neg}}$ is countably infinite, and so also is countable. QED.

Sec 7.4 #11:

$S = \{x \in \mathbb{R} \mid 0 < x < 1\}$

$V = \{x \in \mathbb{R} \mid 2 < x < 5\}$

To Prove: S and V have the same cardinality.

Proof:
Define function $f: S \to V$ as follows:
FOR ALL $x \in S$, $f(x) = 3x + 2$.

This is well-defined, because if $x \in S$, then
$0 < x < 1$ and so $0 < 3x < 3$, and
so $2 < (3x+2) < 5$, and so $(3x+2) \in V$.

[f is one-to-one:] Suppose $u, v \in S$ are such that $f(u) = f(v)$.
Then $3u + 2 = 3v + 2$, $\therefore 3u = 3v$,
$\therefore u = v$. $\therefore$ f is one-to-one.

[f is onto:] Suppose y is any element in V.
Then $2 < y < 5$. Let $x = \frac{1}{3}(y-2)$.
[I must show that $x \in S$.]
Since $2 < y < 5$, $0 < y - 2 < 3$.
$\therefore 0 < \frac{1}{3}(y-2) < 1$. $\therefore 0 < x < 1$. $\therefore x \in S$.
Now, $f(x) = 3x + 2 = 3(\frac{1}{3}(y-2)) + 2$
$\therefore f(x) = (y-2) + 2 = y$. $\therefore$ f is onto.
$\therefore$ f is a one-to-one correspondence.
$\therefore$ S and V have the same cardinality. QED

SEC 7.4 #11, ALTERNATE SOLUTION
USING THEOREM (NIB) 10.

To Prove: $S = (0,1)$ and $V = (2,5)$ have the same cardinality.

Proof: Define function $f: S \to V$ as follows:
For all $x \in S$, $f(x) = 3x + 2$.

Function f is well-defined because, if $x \in S$, then $0 < x < 1$, and so, $0 < 3x < 3$, and so, $2 < 3x + 2 < 5$, and so, $0 < f(x) < 5$, and so $f(x) \in V = (2,5)$.

Define function $g: V \to S$ as follows:
For all $y \in V = (2,5)$, $g(y) = \frac{1}{3}(y-2)$.

It can similarly be shown that g is also well-defined.

Let $x \in S = (0,1)$ be given.
$g \circ f(x) = g(f(x)) = g(3x+2) = \frac{1}{3}((3x+2)-2)$
$= \frac{1}{3}(3x) = x = i_S(x)$.
$\therefore g \circ f = i_S$.

Let $y \in V = (2,5)$ be given. $f \circ g(y) = 3(\frac{1}{3}(y-2)) + 2$
$= (y-2) + 2 = y = i_V(y)$. $\therefore f \circ g = i_V$.
$\therefore$ By THM (NIB) 10, f is a one-to-one correspondence from S to V.
$\therefore$ S and V have the same cardinality. QED.

Sec 7.4

#20 Define $f: \mathbb{Z} \to \mathbb{Z}$ as follows:
For all $n \in \mathbb{Z}$, define $f(n) = 2n$

Define $g: \mathbb{Z} \to \mathbb{Z}$ as follows:
For all $n \in \mathbb{Z}$,

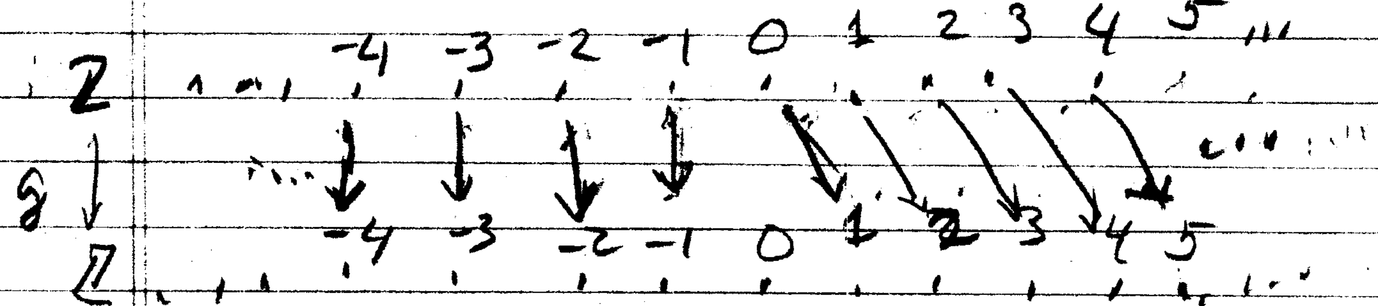
$$\text{define } g(n) = \begin{cases} n+1 & \text{if } n \geq 0 \\ n & \text{if } n < 0 \end{cases}$$

Both f and g are one-to-one.
f is not onto because, for example, $f(n) \neq 1$ for all $n \in \mathbb{Z}$. In fact, the range of f is the set of even integers, not the set of all integers.

g is not onto because $g(n) \neq 0$ for all $n \in \mathbb{Z}$. Thus, for $y = 0$, there does not exist an element n in $\mathbb{Z}$ such that $g(n) = 0$.

There are several other correct solutions.

The arrow diagram for g:



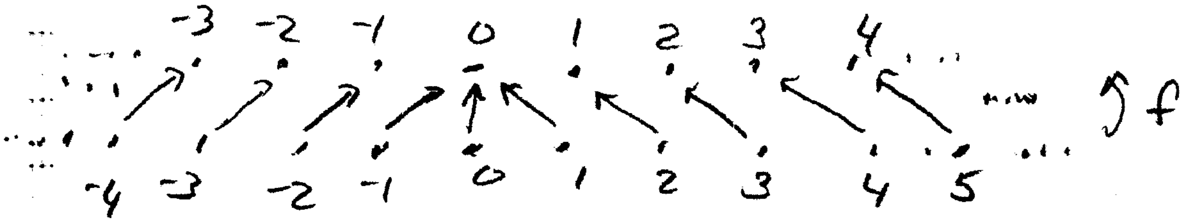
Sec 7.4, #21:

#21 Define $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ as follows:

For all $n \in \mathbb{Z}$,

define $f(n) = \begin{cases} n-1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ n+1 & \text{if } n < 0 \end{cases}$

f is onto, but f is not one-to-one since $f(1) = f(0) = f(-1) = 0$, but $0 \neq 1$ and $0 \neq -1$.

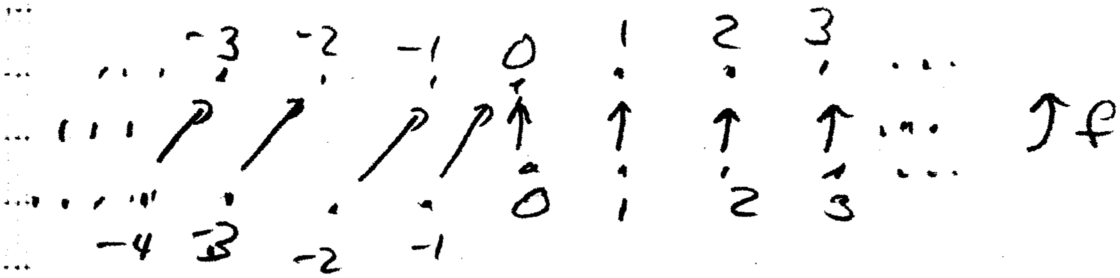


For all $n \in \mathbb{Z}$,

define $g(n) = \begin{cases} n & \text{if } n \geq 0 \\ n+1 & \text{if } n < 0 \end{cases}$

g is onto, but g is not one-to-one since $g(0) = g(-1) = 0$ but $0 \neq -1$.

$f \neq g$ since $f(1) \neq g(1)$



#22 (sec 7.4)

Define $g: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by the formula

$$g((m,n)) = 2^m \cdot 3^n$$

for all $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

We show that g is one-to-one:

Suppose (m_0, n_0) and (m_1, n_1) are elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$. Suppose that

$$g((m_0, n_0)) = g((m_1, n_1)).$$

Then $2^{m_0} \cdot 3^{n_0} = 2^{m_1} \cdot 3^{n_1}$.

Let $k = 2^{m_0} \cdot 3^{n_0}$.

By the Unique Factorization Theorem 4.3.5, any two prime factorizations of k are the same (essentially).

$\therefore$ Since $k = 2^{m_1} \cdot 3^{n_1} = 2^{m_0} \cdot 3^{n_0}$ where m_0, n_0, m_1, n_1 are all positive integers, we can conclude that $m_0 = m_1$ and $n_0 = n_1$ by the UNIQUE FACTORIZATION THM.

$\therefore (m_0, n_0) = (m_1, n_1)$.

$\therefore g$ is one-to-one.

Now, ANY set which has the same cardinality as a countable set is itself a countable set.

Sec 7.4

#22 (cont.)

Let $Y = \{ t \in \mathbb{Z}^+ \mid t = g(x)$ for some $x \in \mathbb{Z}^+ \times \mathbb{Z}^+ \}$

Y is the RANGE OF g, contained in $\mathbb{Z}^+$

Define $h: \mathbb{Z}^+ \times \mathbb{Z}^+ \to Y$ as follows:

For all $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$,
$h((m,n)) = g((m,n))$, that is

$h((m,n)) = 2^m \cdot 3^n$.

The proof that h is one-to-one is the same as the proof that g is one-to-one.
Since Y = RANGE of g, h is clearly onto:
Given $y_0 \in Y$, $y_0 = g(x_0)$ for some $x_0 \in \mathbb{Z}^+ \times \mathbb{Z}^+$.
So $h(x_0) = g(x_0) = y_0$. $\therefore$ h is onto.

$\therefore$ h is a one-to-one correspondence

Now, $\mathbb{Z}^+$ is countable and $Y \subseteq \mathbb{Z}^+$.
By Theorem 7.4.3, Y is countable.
Because h exists, $\mathbb{Z}^+ \times \mathbb{Z}^+$ and Y have the same cardinality.
$\therefore$ $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

QED.