M325K

SPRING 2024

HW #14, Part I, Sec 7.4 Solutions

Sec 7.4, (The solution to #13 is on page 2)

#14: The function $g: \mathbb{R} \to (0,1)$ is a one-to-one correspondence.

From this fact, we can conclude that the interval $(0,1)$ and all of $\mathbb{R}$ have the same cardinality.

The solution for Sec 7.4, #13, appears on the next page.

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Section 7.4 Solutions

and n_2, if $f(n_1) = f(n_2)$ then $3n_1 = 3n_2$ and so $n_1 = n_2$. Also f is onto because if m is any element in $3\mathbf{Z}$, then $m = 3k$ for some integer k. But then $f(k) = 3k = m$ by definition of f. Thus, since there is a function $f: \mathbf{Z} \to 3\mathbf{Z}$ that is one-to-one and onto, $\mathbf{Z}$ has the same cardinality as $3\mathbf{Z}$.

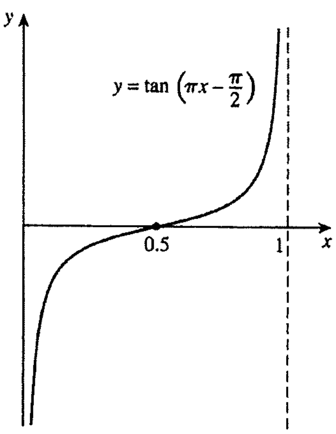
7. *Hint:* If $m \in \mathbf{Z}^+$, show that $j(m) = j(m+1) = m$.

8. It was shown in Example 7.4.2 that $\mathbf{Z}$ is countably infinite, which means that $\mathbf{Z}^+$ has the same cardinality as $\mathbf{Z}$. By exercise 3, $\mathbf{Z}$ has the same cardinality as $3\mathbf{Z}$. It follows by the transitive property of cardinality (Theorem 7.4.1 (c)) that $\mathbf{Z}^+$ has the same cardinality as $3\mathbf{Z}$. Thus $3\mathbf{Z}$ is countably infinite *[by definition of countably infinite]*, and hence $3\mathbf{Z}$ is countable *[by definition of countable]*.

10. Proof: Define $f: S \to U$ by the rule $f(x) = 2x$ for all real numbers x in S. Then f is one-to-one by the same argument as in exercise 10a of Section 7.2 with $\mathbf{R}$ in place of $\mathbf{Z}$. Furthermore, f is onto because if y is any element in U, then $0 < y < 2$ and so $0 < y/2 < 1$. Consequently, $y/2 \in S$ and $f(y/2) = 2(y/2) = y$. Hence f is a one-to-one correspondence, and so S and U have the same cardinality.

11. *Hint:* Define $h: S \to V$ as follows: $h(x) = 3x + 2$, for all $x \in S$.

13.



It is clear from the graph that f is one-to-one (since it is increasing) and that the image of f is all of $\mathbf{R}$ (since the lines $x = 0$ and $x = 1$ are vertical asymptotes). Thus S and $\mathbf{R}$ have the same cardinality.

16. In Example 7.4.4 it was shown that there is a one-to-one correspondence from $\mathbf{Z}^+$ to $\mathbf{Q}^+$. This implies that the positive rational numbers can be written as an infinite sequence: $r_1, r_2, r_3, r_4, \ldots$. Now the set $\mathbf{Q}$ of all rational numbers consists of the numbers in this sequence together with 0 and the negative rational numbers: $-r_1, -r_2, -r_3, -r_4, \ldots$. Let $r_0 = 0$. Then the elements of the set of all rational numbers can be "counted" as follows:

$$r_0, r_1, -r_1, r_2, -r_2, r_3, -r_3, r_4, -r_4, \ldots.$$

In other words, we can define a one-to-one correspondence

$$G(n) = \begin{cases} r_{n/2} & \text{if } n \text{ is even} \\ -r_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases} \quad \text{for all integers } n \geq 1.$$

Therefore, $\mathbf{Q}$ is countably infinite and hence countable.

18. *Hint:* No.

19. *Hint:* Suppose r and s are real numbers with $s > r > 0$. Let n be an integer such that $n > \frac{\sqrt{2}}{s-r}$. Then $s - r > \frac{\sqrt{2}}{n}$. Let $m = \left\lfloor \frac{nr}{\sqrt{2}} \right\rfloor + 1$. Then $m > \frac{nr}{\sqrt{2}} \geq m - 1$. Use the fact that $s = r + (s - r)$ to show that $r < \frac{\sqrt{2}m}{n} < s$.

22. *Hint:* Use the unique factorization of integers theorem (Theorem 4.3.5) and Theorem 7.4.3.

23. a. Define a function $G: \mathbf{Z}^{nonneg} \to \mathbf{Z}^{nonneg} \times \mathbf{Z}^{nonneg}$ as follows: Let $G(0) = (0, 0)$, and then follow the arrows in the diagram, letting each successive ordered pair of integers be the value of G for the next successive integer. Thus, for instance, $G(1) = (1, 0)$, $G(2) = (0, 1)$, $G(3) = (2, 0)$, $G(4) = (1, 1)$, $G(5) = (0, 2)$, $G(6) = (3, 0)$, $G(7) = (2, 1)$, $G(8) = (1, 2)$, and so forth.

    b. *Hint:* Observe that if the top ordered pair of any given diagonal is $(k, 0)$, the entire diagonal (moving from top to bottom) consists of $(k, 0), (k-1, 1), (k-2, 2), \ldots, (2, k-2), (1, k-1), (0, k)$. Thus for all the ordered pairs (m, n) within any given diagonal, the value of $m + n$ is constant, and as you move down the ordered pairs in the diagonal, starting at the top, the value of the second element of the pair keeps increasing by 1.

25. *Hint:* There are at least two different approaches to this problem. One is to use the method discussed in Section 4.2. Another is to suppose that $1.999999\ldots < 2$ and derive a contradiction. (Show that the difference between 2 and $1.999999\ldots$ can be made smaller than any given positive number.)

26. Proof: Let A be an infinite set. Construct a countably infinite subset $a_1, a_2, a_3, \ldots$ of A, by letting a_1 be any element of A, letting a_2 be any element of A other than a_1, letting a_3 be any element of A other than a_1 or a_2, and so forth. This process never stops (and hence $a_1, a_2, a_3, \ldots$ is an infinite sequence) because A is an infinite set. More formally,

    1. Let a_1 be any element of A.
    2. For each integer $n \geq 2$, let a_n be any element of $A - \{a_1, a_2, a_3, \ldots, a_{n-1}\}$. Such an element exists, for otherwise $A - \{a_1, a_2, a_3, \ldots, a_{n-1}\}$ would be empty and A would be finite.

27. Proof: Suppose A is any countably infinite set, B is any set, and $g: A \to B$ is onto. Since A is countably infinite, there is a one-to-one correspondence $f: \mathbf{Z}^+ \to A$. Then, in particular, f is onto, and so by Theorem 7.3.4, $g \circ f$ is an onto function from $\mathbf{Z}^+$ to B. Define a function $h: B \to \mathbf{Z}^+$ as follows: Suppose x is any element of B. Since $g \circ f$ is onto, $\{m \in \mathbf{Z}^+ \mid (g \circ f)(m) = x\} \neq \emptyset$. Thus, by the well-ordering principle for the integers, this set has a least element. In other words, there is a least positive integer n with $(g \circ f)(n) = x$. Let $h(x)$ be this integer.

    We claim that h is a one-to-one. For suppose $h(x_1) = h(x_2) = n$. By definition of h, n is the least positive integer with $(g \circ f)(n) = x_1$. But also by definition of h, n is the least positive integer with $(g \circ f)(n) = x_2$. Hence $x_1 = (g \circ f)(n) = x_2$.

HW #14 SOLUTIONS (cont.)

Part II (i) Placing the elements of the set of rational numbers, $\mathbb{Q}$, on a lattice quadrant:

I will use the following system:

at every point in Row 1, I will place the number 0.

at every point in Row 2, I will place $\frac{n}{1}$, where n = the column # of the column containing that point.

At every point in Row 3, I will place $\frac{n}{-1}$, where n = the column # of the column containing that point.

In Row 4, $\frac{n}{2}$; In Row 5, $\frac{n}{-2}$; in Row 6, $\frac{n}{3}$; in Row 7, $\frac{n}{-3}$, ...

Thus, for each integer $k \geq 2$, at every point in Row k, I will place

$$\begin{cases} \dfrac{n}{\frac{1}{2}(k)} & \text{if } k \text{ is even} \\[2ex] \dfrac{n}{(-\frac{1}{2})(k-1)} & \text{if } k \text{ is odd} \end{cases}$$

where n is the column # of the column containing that point.

START Visually, the result is as follows:

Row 1:	0	0	0	0	0	0	0	...
Row 2:	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	$6/1$	$7/1$	...
Row 3:	$1/-1$	$2/-1$	$3/-1$	$4/-1$	$5/-1$	$6/-1$	$7/-1$	...
Row 4:	$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	$6/2$	$7/2$	...
Row 5:	$1/-2$	$2/-2$	$3/-2$	$4/-2$	$5/-2$	$6/-2$	$7/-2$	...
Row 6:	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	$6/3$	$7/3$	...
Row 7:	$1/-3$	$2/-3$	$3/-3$	$4/-3$	$5/-3$	$6/-3$	$7/-3$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Part II (continued)

(ii) Defining function f:

Using the placement of the rational numbers in $\mathbb{Q}$ on the lattice quadrant described above,

define function $f: \mathbb{Z}^+ \to \mathbb{Q}$ as follows:

For all $n \in \mathbb{Z}^+$,

$f(n)$ = the n^{th} newly encountered rational number, which is encountered along the path indicated through the lattice.

Function f is one-to-one because previously encountered rational numbers are not counted.

Function f is onto because every rational number is eventually encountered along the path for the first time.

Part III: (i) Placing the ordered pairs in the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ on a lattice quadrant:

I will use the following system:

At every point in Row 1, I will place the ordered pair $(n, 1)$, where n is the column # of the column containing that point.

At every point in Row 2, I will place the ordered pair $(n, 2)$, where n is the column # of the column containing that point.

In Row 3, $(n, 3)$; In Row 4, $(n, 4)$; ...

Thus, for each integer $k \geq 1$, at every point in Row k, I will place the ordered pair (k, n) where n is the column # of the column containing that point.

Visually, the result is as follows:

START

Row 1:	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	...
Row 2:	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	...
Row 3:	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	...
Row 4:	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Part III: (ii) Defining function f:

Using the placement of the ordered pairs in $\mathbb{Z}^+ \times \mathbb{Z}^+$ on the lattice quadrant described above,
define function $f: \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ as follows:

For all $n \in \mathbb{Z}^+$,
$f(n)$ = the n^{th} (newly encountered) ordered pair from $\mathbb{Z}^+ \times \mathbb{Z}^+$, which is encountered along the path indicated through the lattice.

Function f is one-to-one because previously encountered ordered pairs are not counted.

Function f is onto because every ordered pair in $\mathbb{Z}^+ \times \mathbb{Z}^+$ is eventually encountered for the first time

The phrases put in parentheses above were put in parentheses because they are actually unnecessary in this case.
This is because, in the placement of ordered pairs described above, no ordered pair appears more than once on the lattice.

PART IV: To PROVE: The set of all irrational numbers is an uncountable set of real numbers.

Proof: Let I = the set of all irrational real numbers. By definition of the term "irrational number", $\mathbb{R} = I \cup \mathbb{Q}$. Now, $(0,1)$ is uncountable by Theorem (N13) 15.

(*) Since $(0,1) \subseteq \mathbb{R}$, $\mathbb{R}$ is uncountable by Corollary 7.4.4.

It was shown in Part II of this homework assignment, that the set of all rational numbers $\mathbb{Q}$ is countable.

Suppose, by way of contradiction, that I is countable. Since $\mathbb{R} = I \cup \mathbb{Q}$ and since I and $\mathbb{Q}$ are both countable, $\mathbb{R}$ is countable, by Theorem (N13) 12, which states "The union of two countable sets is countable.

$\therefore$ $\mathbb{R}$ is countable and $\mathbb{R}$ is uncountable, which is a contradiction.

$\therefore$ I is uncountable, by proof-by-contradiction.

QED.

(*) If the assignment disallows the use of Corollary 7.4.4, then you can cite an earlier problem from this same HW assignment, in which it is shown that the function $f: (0,1) \to \mathbb{R}$, given by $f: x \to \tan(\pi x - \frac{\pi}{2})$, is a one-to-one correspondence, so $\mathbb{R}$ and $(0,1)$ have the same cardinality. Since $(0,1)$ is uncountable, $\mathbb{R}$ is also uncountable.