HW #4A SOLUTIONS, M325K SPRING 2024

PART I OF HW #4A SOLUTION

Section 4.1, #28

To Prove: For all integers n, if n is odd, then n^2 is odd.

Proof: Let n be any integer.

Suppose n is odd. [NTS: n^2 is odd]

$\therefore$, by definition of "odd", there exists an integer k such that $n = 2k+1$.

$$\begin{aligned} \therefore n^2 &= (2k+1)^2 && \text{, by substitution,} \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 && \text{, by rules of algebra.} \end{aligned}$$

Let $t = 2k^2+2k$, which is an integer since sums and products of integers are integers.

$\therefore n^2 = 2t+1$, by substitution.

$\therefore n^2$ is odd, by definition of "odd."

$\therefore$ For all integers n, if n is odd, then n^2 is odd, by Direct Proof.

Q.E.D.

Part II of HW#4A SOLUTION

To Prove: $(-400 \text{ mod } 23) = 14$

<u>Proof:</u> By the Q-R Theorem, there exist unique integers q and r such that

$$-400 = 23q + r \text{ and } 0 \le r < 23.$$

Also, by definition of $(n \text{ mod } d)$, $(-400 \text{ mod } 23) = r$.

Now, $-400 = 23 * (-18) + 14$
and $0 \le 14 < 23$.

So, by the uniqueness of q and r,
$q = -18$ and $r = 14$.

$\therefore$ By substitution, $(-400 \text{ mod } 23) = 14$.

QED.

PART III of HW#4A SOLUTION

Sec 4.4, #25.

To Prove: For all integers a and b, if (a mod 7) = 5 and (b mod 7) = 6, Then (ab mod 7) = 2.

Proof: Let a and b be any integers such that (a mod 7) = 5 and (b mod 7) = 6.

By the Q-R Theorem, there exist unique integers q_1, r_1, q_2, r_2, and q_3, r_3 such that

$a = 7q_1 + r_1$ and $0 \le r_1 < 7$ and

$b = 7q_2 + r_2$ and $0 \le r_2 < 7$ and

$ab = 7q_3 + r_3$ and $0 \le r_3 < 7$.

By definition of "(n mod d)", (ab mod 7) = r_3,
r_1 = (a mod 7) = 5 and r_2 = (b mod 7) = 6.

$\therefore a = 7q_1 + 5$ and $b = 7q_2 + 6$, by substitution.

$\therefore ab = (7q_1 + 5)(7q_2 + 6)$, by substitution,

$= 49q_1q_2 + 35q_2 + 42q_1 + 30$

$= 49q_1q_2 + 35q_2 + 42q_1 + 28 + 2$

$= 7(7q_1q_2 + 5q_2 + 6q_1 + 4) + 2$, by Rules of Algebra

Let $t = (7q_1q_2 + 5q_2 + 6q_1 + 4)$, which is an integer since sums and products of integers are integers.

$\therefore ab = 7t + 2$, by substitution.

Thus t and 2 are integers such that
$ab = 7t + 2$ and $0 \leq 2 < 7$.

By uniqueness, $q_3 = t$ and $r_3 = 2$.

As shown above, $(ab \bmod 7) = r_3$.

$\therefore (ab \bmod 7) = 2$.

$\therefore$ For all integers a and b, if $(a \bmod 7) = 5$ and $(b \bmod 7) = 6$, then $(ab \bmod 7) = 2$, by Direct Proof.

QED