M 325K SPRING 2024

Hw #4, Sec. 4.4 Solutions

(NOT ASSIGNED) #3 $n = 36$, $d = 40$

$$\begin{array}{r} 0 = q \\ 40\overline{)36} \\ -0 \\ \hline 36 = r \end{array}$$

Since $36 = 40 \times 0 + 36$
and $0 \le 36 < 40$,

$\underline{q = 0 \text{ and } r = 36.}$

#4 $n = 3$ and $d = 11$

$$\begin{array}{r} 0 = q \\ 11\overline{)3} \\ -0 \\ \hline 3 = r \end{array}$$

Since $3 = 11 \times 0 + 3$
and $0 \le 3 < 11$,

$\underline{q = 0 \text{ and } r = 3}$.

#5 $n = -45$; $d = 11$

$11 \times (-4) = -44$; so, $-45 = 11 \times (-4) + (-1)$
but $-1 < 0$, so $q \ne -4$.

$11 \times (-5) = -55$.

So, $-45 = 11 \times (-5) + 10$ and $0 \le 10 < 11$,

so $\underline{q = -5 \text{ and } r = 10}$

Sec 4.4, #6. $n = -27$, $d = 8$

$-32 + 5 = -27$, so $\quad -27 = 8 \times (-4) + 5$

and $0 \le 5 < 8$

so $q = -4$ and $r = 5$.

#7. Here, $n = 43$ and $d = 9$.

$$9 \overline{)43} \quad 4 = q$$
$$\underline{36}$$
$$7 = r$$

$43 = 9 \times 4 + 7$ and $0 \le 7 < 9$,

so, $\boxed{(43 \text{ div } 9) = 4 \text{ and } (43 \text{ mod } 9) = 7}$

#8. Here, $n = 50$ and $d = 7$.

$$7 \overline{)50} \quad 7$$
$$\underline{-49}$$
$$1$$

$50 = 7 \times 7 + 1$ and $0 \le 1 < 7$,

so, $\boxed{(50 \text{ div } 7) = 7 \text{ and } (50 \text{ mod } 7) = 1}$.

Sec 4.4,
\# 21

If $(b \bmod 12) = 5$, then $(8b \bmod 12) = 4$.

To Prove: For every integer b, if $(b \bmod 12) = 5$, then $(8b \bmod 12) = 4$.

Proof: Let b be any integer. Suppose that $(b \bmod 12) = 5$.
[NTS: $(8b \bmod 12) = 4$]

By Quotient-Remainder Theorem,
There exist unique integers q_1, r_1 and q_2, r_2 such that

① $b = 12q_1 + r_1$ and $0 \le r_1 < 12$, and

② $8b = 12q_2 + r_2$ and $0 \le r_2 < 12$.

By the def'n of "$(n \bmod d)$"
$(b \bmod 12) = r_1$ and $(8b \bmod 12) = r_2$.

By substitution, $r_1 = 5$
and $b = 12q_1 + 5$.

$$\begin{aligned}\therefore 8b &= 8(12q_1 + 5), \text{ by substitution}\\ &= 8 \times 12q_1 + 40\\ &= 12(8q_1) + 12 \times 3 + 4\\ &= 12(8q_1 + 3) + 4\\ &= 12t + 4, \text{ where } t = (8q_1 + 3), \text{ by R.O.A.}\end{aligned}$$

and t is an integer since sums and products of integers are integers

$\therefore 8b = 12t + 4$ and $0 \le 4 < 12$.

$\therefore$ By uniqueness, $q_3 = t$ and $r_3 = 4$.

As shown above, $r_3 = (8b \bmod 12)$.

$\therefore (8b \bmod 12) = 4$, by substitution.

$\therefore$ For every integer b, if $(b \bmod 12) = 5$, then $(8b \bmod 12) = 4$, by Direct Proof.

QED.

Sec 4.4, #25. (NOT ASSIGNED)

To Prove: For all integers a and b, if $(a \bmod 7) = 5$ and $(b \bmod 7) = 6$, Then $(ab \bmod 7) = 2$.

Proof: Let a and b be any integers such that $(a \bmod 7) = 5$ and $(b \bmod 7) = 6$.

By the Q-R Theorem, There exist unique integers q_1, r_1, q_2, r_2, and q_3, r_3 such That

$a = 7q_1 + r_1$ and $0 \le r_1 < 7$ and

$b = 7q_2 + r_2$ and $0 \le r_2 < 7$ and

$ab = 7q_3 + r_3$ and $0 \le r_3 < 7$.

By definition of "$(n \bmod d)$", $(ab \bmod 7) = r_3$, $r_1 = (a \bmod 7) = 5$ and $r_2 = (b \bmod 7) = 6$.

$\therefore$ $a = 7q_1 + 5$ and $b = 7q_2 + 6$, by substitution.

$\therefore$ $ab = (7q_1 + 5)(7q_2 + 6)$, by substitution,

$= 49q_1q_2 + 35q_2 + 42q_1 + 30$

$= 49q_1q_2 + 35q_2 + 42q_1 + 28 + 2$

$= 7(7q_1q_2 + 5q_2 + 6q_1 + 4) + 2$, by Rules of Algebra

Let $t = (7q_1q_2 + 5q_2 + 6q_1 + 4)$, which is an integer since sums and products of integers are integers.

$\therefore ab = 7t + 2$, by substitution.

Thus t and 2 are integers such that
$ab = 7t + 2$ and $0 \leq 2 < 7$.

By uniqueness, $q_3 = t$ and $r_3 = 2$.

As shown above, $(ab \bmod 7) = r_3$.

$\therefore (ab \bmod 7) = 2$.

$\therefore$ For all integers a and b, if $(a \bmod 7) = 5$ and $(b \bmod 7) = 6$, then $(ab \bmod 7) = 2$, by Direct Proof.

QED